

ALMOST CONTACT 5-FOLDS ARE CONTACT

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ABSTRACT. We prove that every homotopy class of an almost contact structure on a closed 5-dimensional manifold with no 2-torsion in $H^2(M, \mathbb{Z})$ admits a contact structure. If there is 2-torsion, we prove that there exists a contact structure with the same first Chern class as the given almost contact structure.

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Date: March, 2012.

1991 *Mathematics Subject Classification.* Primary: 53D10. Secondary: 53D15, 57R17.

Key words and phrases. contact structures, Lefschetz pencils.

First author was financially supported by a research grant from La Caixa. Second author would like to thank ICTP for offering a visiting position in 2011 that allowed him to develop this article. Third author was supported by the Spanish National Research Project MTM2010-17389.

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1. INTRODUCTION

Let (M^{2n+1}, ξ) be a cooriented contact manifold, with contact form α . This structure determines a symplectic distribution $(\xi, d\alpha|_\xi) \subset TM$. Any change of the contact form α does not change the conformal symplectic class of $d\alpha$ restricted to ξ . This allows us to choose a compatible almost complex structure $J \in \text{End}(\xi)$. Thus given a cooriented contact structure we obtain in a natural way a reduction of the structure group $Gl(2n+1, \mathbb{R})$ of the tangent bundle TM to the group $U(n) \times \{1\}$, which is unique up to homotopy. We say that a manifold M is an *almost contact manifold* if the structure group of its tangent bundle can be reduced to $U(n) \times \{1\}$. In particular, cooriented contact manifolds are always almost contact and such a reduction of the structure group of the tangent bundle of a manifold M is a necessary condition for the existence of a cooriented contact structure on M . It is unknown whether this condition is in general sufficient.

There are well known cases where having an almost contact structure is sufficient for the manifold to be contact. For example, if the manifold M is open then one can apply Gromov's h -principle techniques to conclude that the condition is sufficient ([EM], 10.3.2). The scenario is quite different for closed almost contact manifolds. Using results of Lutz [Lu] and Martinet [Ma] one can show that every cooriented tangent 2-plane field on a closed, oriented 3-manifold is homotopic to a contact structure. A good account of this result from a modern perspective is given in [Ge]. For manifolds of higher dimensions there are various results establishing the sufficiency of the condition. Important instances of these are the construction of contact structures on certain principal S^1 -bundles over closed symplectic manifolds due to Boothby and Wang [BW], the existence of a contact structure on the product of a contact manifold with a surface of genus greater than zero following Bourgeois [Bo] and the existence of contact structures on simply connected 5-dimensional orientable closed manifolds obtained by Geiges [Ge1] and its higher dimensional analogue [Ge2].

We now turn our attention to 5-manifolds since the main goal of this article is to show that any orientable almost contact 5-manifold is contact. In this case H. Geiges has been studying existence results in other situations apart from the simply connected one, in [GT1] a positive answer is also given for spin closed manifolds with $\pi_1 = \mathbb{Z}_2$ and spin closed manifolds with finite fundamental group of odd order are studied in [GT2]. On the

other hand there is also a construction of contact structures on an orientable 5-manifold occurring as a product of two lower dimensional manifolds by Geiges and Stipsicz [GS]. While Geiges used the topological classification of simply connected manifolds for his results in [Ge1], one of the key ingredients in [GS] is a decomposition result of a 4-manifold into two Stein manifolds with common contact boundary [AM], [Bk].

Being almost contact is a purely topological condition. In fact, the reduction of the structure group can be studied via obstruction theory. For example, in the 5-dimensional situation a manifold M is almost contact if and only if the third integral Steifel–Whitney class $W_3(M)$ vanishes. Actually, using this hypothesis and the classification of simply connected manifolds due to D. Barden [Ba], H. Geiges deduces that any manifold with $W_3(M) = 0$ can be obtained by Legendrian surgery from certain model contact manifolds. Though this approach is elegant, it seems quite difficult to extend these ideas to produce contact structures on any almost contact 5-manifold. We therefore propose a different approach where the required topological property to produce the contact structure is the existence of an *almost contact pencil* structure on the given almost contact manifold. The tools required for the proof using our techniques come from three different sources:

- (i) The approximately holomorphic techniques developed by Donaldson in the symplectic setting [Do, Do1] and adapted in [IMP] to the contact setting to produce the so-called *quasi contact pencil*.
- (ii) The generalization of the notion of overtwistedness to higher dimensions done by Niederkrüger and Gromov [Ni] and the generalized Lutz twist based on that defined in [EP].
- (iii) Eliashberg’s classification of overtwisted 3-dimensional manifolds [El] to produce overtwisted contact structures on the fibres of the pencil.

We state our main result:

Theorem 1.1. *Let M be a closed oriented 5-dimensional manifold with no 2-torsion in $H^2(M, \mathbb{Z})$. There exists a contact structure in every homotopy class of almost contact structures.*

In the general case we prove

Theorem 1.2. *Let M be a closed oriented 5-dimensional manifold. Given an almost contact structure ξ with first Chern class $c_1(\xi)$, there exists a contact structure ξ' such that $c_1(\xi') = c_1(\xi)$.*

In particular closed oriented almost contact 5-manifolds are contact. The two theorems follow the same pattern: we are able to produce a contact structure for any given first Chern class. The difference being that the first Chern class completely determines the homotopy type of the distribution if there are no 2-torsion elements in the second cohomology group. This will be carefully explained in the proof.

It is important to remark that using the techniques developed in this article, it is not possible to say anything about the number of distinct contact distributions that may occur in a given homotopy class of almost contact distributions. The result states that there is at least one. From the construction it will be clear that the contact structure contains a PS-structure [Ni, NP] and therefore it is non-fillable. Actually, the PS-structure is used as a substitute of the overtwisted disk to make the structure flexible as Eliashberg does in the 3-dimensional case.

Remark 1.3. *We assume from now on that the distributions are coorientable. Section 10 contains the corresponding results for non-coorientable distributions. In particular, we will state and prove the equivalent of Theorem 1.2 in that setting.*

The two main results essentially have the same proof. It is a constructive argument in which we obtain the contact condition step by step. These steps roughly correspond to the sections of the paper as follows:

- We first show that any almost contact 5-manifold (M, ξ) admits an almost contact fibration over \mathbb{S}^2 with singularities of some standard type. The construction of this *almost contact fibration* – actually an almost contact pencil – is the content of Sections 2 and 3.
- In Section 4, we produce a first perturbation of the almost contact structure ξ to make it contact in the neighborhoods of the singularities of the fibration.
- One of the types of singularities has the structure of a *base locus* of a pencil occurring in algebraic or symplectic geometry. So, to provide a Lefschetz type fibration we need to blow-up the *base locus*. For that we need to define the notion of contact blow-up. For the purposes of the article, it will be enough to define the contact blow-up of a contact 5-manifold along a transverse \mathbb{S}^1 and the corresponding notion of contact blow-down. This is the content of Section 5.
- Away from the critical points the distribution splits as $\xi = \xi_v \oplus \mathcal{H}$, where ξ_v is the restriction of the distribution to the fibres and \mathcal{H} is the symplectic orthogonal. Section 6 deals with a deformation of ξ_v to make the fibres contact submanifolds. It strongly uses the

construction of overtwisted contact manifolds due to Eliashberg [El].

- In Section 7 we deform the other direction \mathcal{H} . This is done in two steps. We take a suitable cell decomposition of \mathbb{S}^2 and we deform in the fibres of the preimage of a neighborhood of the 1-skeleton.
- We are left with filling the preimage of the 2-cells. This is done by means of a *standard model* precisely constructed in Section 8.
- Gathering the results in the previous sections we obtain the desired contact structure in Section 9.
- In Section 10 we deal with the case of non-coorientable distributions. First, we introduce the suitable definitions and afterwards we state and prove the non-coorientable version of Theorem 1.2.

It is reasonable to guess after a careful reading of this article what is needed to adapt the proofs in order to work in the 7-dimensional case. We have begun to understand this 7-dimensional setup and it will be the goal of a forthcoming article. E. Giroux has work in progress in which he tries to prove these existence results by using an open book decomposition of the manifold [Gi].

2. QUASI-CONTACT STRUCTURES.

Let M be an almost contact manifold. It is easy to see that there is always a choice of symplectic distribution $(\xi, \omega) \subset TM$ for such a manifold. Namely, we can find a 2-form η on ξ with the property that η is non-degenerate and compatible with the almost complex structure J defined on ξ . By extending η to a form on M we can find a (not necessarily closed) 2-form ω on M such that $(\xi, \omega|_{\xi})$ becomes a symplectic vector bundle. By slight abuse of notation we will say that (M, ξ, ω) is an almost contact manifold. In other words, by an almost contact structure we will always mean a triple (ξ, J, ω) for some ω as discussed. We will always talk about codimension 1 distributions of M which are almost complex and hence from now on we will denote an almost contact manifold by a triple (M, ξ, ω) .

In order to construct a contact structure out of an almost contact one, the first step is to provide a better 2-form on M . That is, we would like to replace ω by a closed 2-form. We begin by defining the notion of a *quasi-contact structure*.

Definition 2.1. *A manifold M^{2n+1} admits a quasi-contact structure if there exists a pair (ξ, ω) such that ξ is a codimension 1-distribution and ω is a closed 2-form on M which is non-degenerate when restricted to ξ .*

Notice that the pair (ξ, ω) always admits a compatible almost contact structure. That is, there exist a J which makes (ξ, J, ω) into an almost

contact structure. We would like to add here that these manifolds have also been called *2-calibrated* [IM] in the literature. The following lemma justifies the appearance of the previous definition:

Lemma 2.2. *Every almost contact manifold (M, ξ_0, ω_0) admits a quasi-contact structure (ξ_1, ω_1) homotopic to (ξ_0, ω_0) through symplectic distributions and the class $[\omega_1]$ can be fixed to be any prescribed cohomology class $a \in H^2(M, \mathbb{R})$.*

Proof. Let $j : M \rightarrow M \times \mathbb{R}$ be the inclusion as the zero section. We know that we can find a (not-necessarily closed) 2-form $\tilde{\omega}_0$, such that $\omega_0 = j^*\tilde{\omega}_0$. Fix a Riemannian metric g over M such that ξ_0 and $\ker \omega_0$ are g -orthogonal.

Apply Gromov's classification result ([EM], Corollary 10.2.2) of open symplectic manifolds to produce a 1-parametric family $\{\tilde{\omega}_t\}_{t=0}^1$ of *symplectic* forms such that for $t = 1$ the form is closed. Moreover choose the cohomology class defined by $\tilde{\omega}_1$ to be π^*a , where the map $\pi : M \times \mathbb{R} \rightarrow M$ is the usual projection. Consider the family of 2-forms $\omega_t = j^*\tilde{\omega}_t$ on M . Since $\tilde{\omega}_t$ is non-degenerate on $M \times \mathbb{R}$ for each t , the form ω_t has 1-dimensional kernel $\ker \omega_t$. Define $\xi_t = (\ker \omega_t)^\perp$, then (ξ_t, ω_t) provides the required family. \square

This is the farthest one can reach by the standard h -principle argument in order to find contact structures on a closed manifold. One can start with the almost contact bundle $\xi = \ker \alpha$ and find a 2-form $d\beta$ that makes it symplectic by Lemma 2.2, but there is in general no way to relate α and β . This is the whole task of this article.

3. QUASI-CONTACT PENCILS.

Approximately holomorphic techniques have been extremely useful in symplectic geometry. Their main application in contact geometry – due to Giroux – is to establish the existence of a compatible open book for a contact manifold in higher dimensions. An open book decomposition is a way of *trivializing* a contact manifold M^{2n+1} by fibering it over \mathbb{S}^1 . Such objects have also been studied in the almost contact case, see [MMP]. Analogously to the Lefschetz pencil decomposition introduced by Donaldson over a symplectic manifold [Do1], there is a construction in the contact case [Pr1] that allows us to express a contact manifold as a (possibly singular) fibration with \mathbb{S}^2 as the base. It has been extended in [IM2] to the quasi-contact setting. We encourage the reader to keep track of the terms *almost contact*, *quasi-contact* and *contact* in the following definitions. It will be important to remember which of these definitions and results apply to each of them.

We begin by introducing the definition of an almost contact (resp. quasi-contact) pencil. Recall that an almost contact (resp. quasi-contact) submanifold of an almost contact (resp. quasi-contact) manifold (M, ξ, ω) is an embedded submanifold $j : S \rightarrow M$ such that the induced pair $(j^*\xi, j^*\omega)$ is almost contact (resp. quasi-contact). In particular this implies that

the submanifold is transverse to the distribution. We say that a chart $\phi_U : (U, p) \rightarrow V \subset (\mathbb{C}^n \times \mathbb{R}, 0)$ is compatible with the almost contact (resp. quasi-contact) structure at a point $p \in U \subset M$ if the push-forward at p of ξ_p is $\mathbb{C}^n \times \{0\}$ and moreover $(\phi_U)_* \omega(p)$ is a positive $(1, 1)$ -form. It is straightforward to give the definition of contact submanifold of a contact manifold in the same spirit.

Definition 3.1. *An almost contact pencil on a closed almost contact manifold (M^{2n+1}, ξ, ω) is a triple (f, B, C) consisting of a codimension-4 almost contact submanifold B , called the base locus, a finite set C of smooth transverse curves and a map $f : M \setminus B \rightarrow \mathbb{CP}^1$ conforming the following conditions:*

- (1) *The set $f(C)$ contains locally smooth curves with transverse self-intersections and the map f is a submersion on the complement of C .*
- (2) *Each $b \in B$ has a compatible local coordinate map to $(\mathbb{C}^n \times \mathbb{R}, 0)$ under which B is locally cut out by $\{z_1 = z_2 = 0\}$ and f corresponds to the projectivization of the first two coordinates, i.e. locally $f(z_1, \dots, z_n, t) = \frac{z_2}{z_1}$.*
- (3) *At a critical point $p \in \gamma \subset M$ there exists a compatible local coordinate chart ϕ_P such that*

$$(f \circ \phi_P^{-1})(z_1, \dots, z_n, s) = f(p) + z_1^2 + \dots + z_n^2 + g(s)$$

where $g : (\mathbb{R}, 0) \rightarrow (\mathbb{C}, 0)$ is a submersion at the origin.

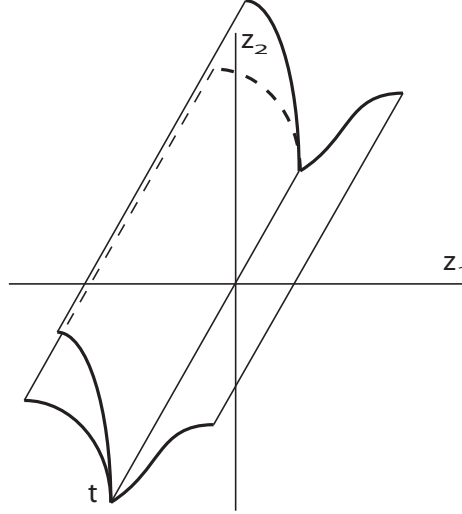
- (4) *The fibres $f^{-1}(P)$, for any $P \in \mathbb{CP}^1$, are almost contact submanifolds at the regular points.*

The local models have to be provided through a compatible coordinate chart in the sense above, that is to say the distribution in $\mathbb{C}^n \times \mathbb{R}$ is mapped to the horizontal distribution $\mathbb{C}^n \times \{0\}$ and ω is a positive $(1, 1)$ -form when restricted to the horizontal distribution with respect to its canonical almost complex structure.

Remark 3.2. *Likewise we can define quasi-contact pencils for quasi-contact manifolds and contact pencils for contact manifolds by replacing the expression almost contact by the suitable one in each case.*

The generic fibres of f are open almost contact submanifolds and the closures of the fibres at the base locus are smooth. This is because the local model (2) in the Definition 3.1 is a parametrized elliptic singularity and the fibres come in complex lines ($z_2 = \text{const} \cdot z_1$) joining at the origin. We refer to the compactified fibres so constructed as the fibres of the pencil. See Figure 1.

Notice that the set of critical values $\Delta = f(C)$ are no longer points, as in the symplectic case, but immersed curves. This is because of Condition

FIGURE 1. fibres close to B

(3) in the Definition 3.1. In particular, the usual isotopy argument between two fibres does not apply unless their images are in the same connected component of $\mathbb{CP}^1 \setminus \Delta$. This has been studied in the contact case and in the general quasi-contact case. Recall that the set C is oriented (since it is a positive link) and therefore Δ is also oriented. There is a partial order in the complementary of Δ defined saying that P_0 is less or equal than P_1 if they can be connected by an oriented path $\gamma \subset \mathbb{CP}^1$ such that intersects Δ only with positive crossings. The proposition that follows has only been proved for the quasi-contact (and contact) case, but it probably remains true in the almost contact setting. It is provided to offer some geometric insight about contact and quasi-contact pencils, it is not used in the rest of the article. Nevertheless, it is a good picture to keep in mind.

Proposition 3.3 (Proposition 6.1 of [Pr1]). *Let M be a quasi-contact manifold equipped with a quasi-contact pencil (f, B, C) . Then if two regular values of f , P_0 and P_1 , are separated by a unique curve of Δ then the two corresponding fibres $F_0 = \overline{f^{-1}(P_0)}$ and $F_1 = \overline{f^{-1}(P_1)}$ are related by an index $n - 1$ surgery. Moreover if the manifold and the pencil are contact, then the surgery is a Legendrian one and it attaches a Legendrian sphere to F_0 if P_0 is smaller than P_1 (and viceversa, see Figure 2).*

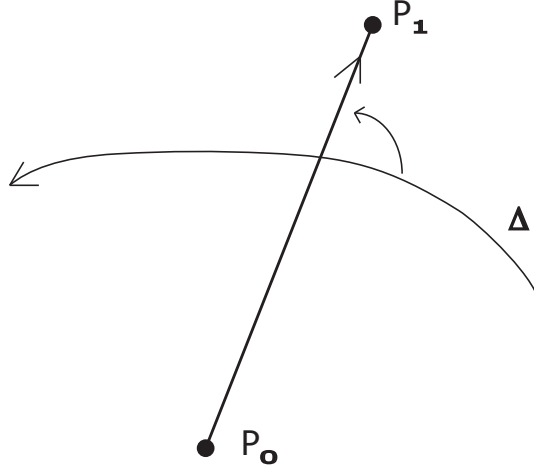


FIGURE 2. The drawn orientations make $F_1 = \overline{f^{-1}(P_1)}$ a Legendrian surgery of $F_0 = \overline{f^{-1}(P_0)}$.

In the contact case we are claiming that the crossing of the singular curve in the fibration amounts to a directed Weinstein cobordism. In the quasi-contact case no such orientation appears. For instance, the case in which the quasi-contact distribution is a foliation (in dimension 3 this is a *taut* foliation) becomes absolutely symmetric and there is no difference in crossing one way or the other.

The main existence result ([IM2, Pr1]) can be stated as

Theorem 3.4. *Let (M, ξ, ω) be a quasi-contact manifold. Given an integral class $a \in H^2(M, \mathbb{Z})$, there exists a quasi-contact pencil (f, B, C) such that the fibres are Poincaré dual to the class $a + k[\omega]$, for some $k \in \mathbb{N}$.*

The proof of this result does not work in the almost contact setting. In order to construct the pencil, the approximately holomorphic techniques are essential and for them to work we need the closedness of the 2-form ω . In general, a quasi-contact pencil may have empty base locus, however a pencil obtained through approximately holomorphic sections over a higher dimensional manifold does not.

In the case the form ω of the quasi-contact structure is exact – then called an *exact quasi-contact structure* – we have the following

Corollary 3.5. *Let (M, ξ, ω) be an exact quasi-contact closed manifold. Then it admits a quasi-contact pencil such that any smooth fibre F satisfies $c_1(\xi_F) = 0$. Further, one can assume that the set B is always non-empty if the dimension of M is greater than 3.*

Proof. We use Theorem 3.4 to construct a pencil such that the cohomology class of the Poincaré dual to the smooth fibers equals the first Chern class $c_1(\xi)$ of the complex bundle ξ . This follows from the fact that for any $a \in H^2(M, \mathbb{Z})$ there is a quasi-contact pencil with any smooth fiber Poincaré dual to the class $a + k[\omega]$. Since ω is exact we get the pencil with the property mentioned earlier by taking $a = c_1(\xi)$.

Since the distribution ξ is transverse to the fibre F , we have that

$$\xi|_F = (\xi|_F \cap TF) \oplus \nu(F).$$

Recall that $c_1(\nu(F)) = e(\nu(F)) = PD([F])$ to obtain

$$c_1(\xi|_F) = c_1(\xi|_F \cap TF) + c_1(\nu(F)) = c_1(\xi_F) + a|_F,$$

and so

$$a|_F = c_1(\xi_F) + a|_F.$$

And this shows that the first Chern class of the quasi-contact structure in any smooth fibre is zero.

As for the non-emptiness of the set B , just recall that by the general theory developed in [IM2, IMP], the submanifold satisfies a Lefschetz hyperplane theorem which in turn implies that whenever the dimension of M is greater than 3, the morphism

$$H_0(B) \longrightarrow H_0(M)$$

is surjective. Hence we conclude that B is not the empty set. \square

The triviality of the Chern class of the quasi-contact structures in the fibres, as well as the base locus being non-empty, will be essential in the construction of the contact structure.

4. DEFORMING THE LOCAL MODELS.

The starting point is a quasi-contact pencil provided by Corollary 3.5. Our task is now to modify the quasi-contact structure by a C^0 -small perturbation in such a way that it becomes contact in a neighborhood of the critical and base point sets B and C of the quasi-contact pencil and almost contact in the complement of them. Being a C^0 -small perturbation, the pencil remains an almost contact pencil adapted to the new almost contact structure. So we will be allowed to work with an almost contact structure and adapted pencil chosen in such a way that the structure is contact close to the sets B and C . This is the first moment of the proof in which the 5-dimensional assumption comes into play.

For the sake of completeness we include a proof of the following lemma.

Lemma 4.1. *Let $E \rightarrow M$ be a vector bundle. Assume that there is a 2-form ω on the total space of E such that restricted to each fibre F is a bilinear symplectic form. Denote by $k_p : F_p \rightarrow E$ the inclusion map of each fibre into the vector bundle. Then, there exists a 1-form β on the manifold E such that $k_p^* d\beta = k_p^* \omega$ and the form β restricted to the zero section of the vector bundle is null.*

Proof. Let $\{U_i\}$ be a covering of M by contractible open sets. There exists a trivializing bundle chart $\Phi_i : E|_{U_i} \rightarrow U_i \times F$ such that $j_p^*(\Phi_i)_* \omega = \omega_0$, where ω_0 is a fixed symplectic form in the vector space F and $j_p : F \rightarrow U_i \times F$ is the inclusion map of the fibre $\{p\} \times F$. There also exists a 1-form β_0 over F such that $\omega_0 = d\beta_0$ and $\beta_0(0) = 0$. Let $\pi_i : U_i \times F \rightarrow F$ be the projection onto the second factor and we construct $\bar{\beta}_0 = \pi_i^* \beta_0$. Therefore over $U_i \times F$, we have that

$$j_p^*(\Phi_i)_* \omega = j_p^*(d\bar{\beta}_0),$$

for every $p \in U_i$. We have that $j_p = \Phi_i \circ k_p$ in U_i and so we obtain

$$k_p^* \omega = k_p^* \Phi_i^*(d\bar{\beta}_0) = k_p^* d\Phi_i^*(\bar{\beta}_0),$$

for every $p \in U_i$. We denote $\beta_i = \Phi_i^* \bar{\beta}_0$. Finally, we fix a partition of the unity $\{\chi_i\}$ compatible with the covering $\{U_i\}$. The global 1-form

$$\beta = \sum_i \chi_i \cdot \beta_i$$

is the required form as χ_i is constant along the fibres. \square

A perturbation of an almost contact pair (ξ, ω) is a 1-parametric family $\{(\xi_t, \omega_t)\}_{t=0}^1$. It is small in C^k -topology if each of the elements of the pair is small in that topology. Given a quasi-contact structure and a quasi-contact pencil adapted to it, the latter will remain adapted to a C^0 -small perturbation of the former.

Definition 4.2. *A submanifold $i : S \rightarrow M$ of an almost contact manifold (M, ξ, ω) is said to be contact if it is almost contact and there is a choice of adapted form α for ξ in a neighborhood of S , i.e $\xi = \ker \alpha$, such that $i^*(d\alpha) = i^* \omega$.*

In particular, a transverse loop is a contact submanifold. This is the case of interest in this article. We have given a more general definition in order to deal with the higher-dimensional situation, this will be the working definition in sequels of this article. For that reason, we show how to perturb in the neighborhood of any contact submanifold to get a contact structure close by.

Proposition 4.3. *Let (M, ξ_0, ω_0) be a quasi-contact manifold and S a contact submanifold of it. There exists a C^0 -small perturbation $\{(\xi_t, \omega_t)\}$ of the pair such that the almost contact structure (ξ_1, ω_1) coincides with $(\xi_1, d\alpha_1)$,*

with α_1 satisfying $\ker \alpha_1 = \xi_1$ in a neighborhood of S . In general, the new structure fails to be quasi-contact.

In other words the quasi-contact structure can be made contact in a neighborhood of the contact submanifold by a slight perturbation at the price of losing the closedness of the 2-form in a larger neighborhood. Let us give the following convention for the forthcoming proof: given a r -form γ on a manifold M and a closed submanifold S , we shall denote by $\gamma|_S$ the restriction of γ to $(\Lambda^r T^*M)|_S$.

Proof. Let $i : S \rightarrow M$ be the inclusion map of the submanifold S in M . Since the submanifold S is quasi-contact, we have that

$$(\xi_0)|_S = \xi_S \oplus \nu(S),$$

where the bundle $\pi : \nu(S) \rightarrow S$ is the ω_0 -orthogonal of the distribution ξ_S inside $(\xi_0)|_S$. Denote by $i_0 : S \rightarrow \nu(S)$ the inclusion of S in $\nu(S)$ as its zero section S_0 . The symplectic form $(\omega_0)|_{\nu(S)}$ is just defined on $\pi^*\nu(S) \subset T\nu(S)$, it can be extended to a whole 2-form in the manifold $\nu(S)$. Since one can choose an affine connection (horizontal distribution) H on the manifold $\nu(S)$, i.e. a splitting $T\nu(S) = \pi^*\nu(S) \oplus H$. Lift $(S, i^*\omega_0)$ using this connection so as to obtain a form (H, ω_H) defined over the distribution H as $\omega_H(h_1, h_2) = \omega_0(d\pi(h_1), d\pi(h_2))$. Define $\omega' = \pi^*((\omega_0)|_{\nu(S)}) \oplus \omega_H$.

The so constructed 2-form ω' is not necessarily closed. It is clear that $(\omega')|_{i_0(S)} = (\omega_0)|_S$. We apply Lemma 4.1 to the pair $(\nu(S), \omega')$ to find a 1-form β' satisfying that $d\beta'$ restricted to the fibres coincides with ω' , also restricted to them. Note that the following equality holds

$$(1) \quad i_v d\beta'(p) = 0, \quad \forall p \in S_0, v \in (i_0)_* T_p S.$$

Indeed, since $i_0^*\beta' = 0$, $i_0^*d\beta' = 0$ and therefore we have $i_w i_v d\beta'(p) = 0$, for vectors $v, w \in T_p S$. As for the *vertical* direction $w \in \nu(S) = (TS)^{\omega_0}$, the equality $i_w i_v d\beta'(p) = 0$ holds by definition of ω_0 -orthogonality.

By using the exponential map of an auxiliary Riemannian metric g we construct an isomorphism $\phi : \nu(S) \rightarrow U \subset M$ between $\nu(S)$ and a tubular neighborhood of S . The metric can be easily chosen in such a way that $(\phi^*\omega)|_{S_0} = \omega'|_{S_0}$. Choose a form α compatible with ξ in a neighborhood of S that makes S a contact submanifold. We have that

$$(2) \quad i^* d\alpha = i^* \omega = i^* \phi^* \omega'$$

To extend over a neighborhood define the following form over $\nu(S)$

$$\alpha_d = \pi^* i^* \alpha + \beta'$$

Note α_d coincides with α over the submanifold $S \subset M$, and further we have

$$d\alpha_d = d\pi^* i^* \alpha + d\beta'.$$

So far, at any given point $p \in S_0$ the following holds:

(i) Because S is a contact submanifold of M :

$$d\pi^*i^*\alpha|_H = d\pi^*i^*\alpha|_{TS_0} = \phi^*\omega|_{TS_0}$$

- (ii) The kernel of the 2-form $d\pi^*i^*\alpha$ contains $V = \pi^*\nu(S)$.
- (iii) $d\beta|_V = \phi_*\omega'_V$, for the 1-form β' is constructed as in Lemma 4.1.
- (iv) The kernel of $d\beta'$ is $H = TS_0$, by equality (1).

Define $\hat{\alpha}_1 = \phi_*\alpha_d$. The previous list of facts proves that over the submanifold S the forms $d\hat{\alpha}_1$ and ω coincide. We have obtained that the form $\hat{\alpha}_1$ is a contact form at points of S . So, since the contact condition is open, $\hat{\alpha}_1$ is also a contact form in a small neighborhood of S .

Let $\chi_\varepsilon : [0, \infty) \rightarrow [0, 1]$ be a decreasing cut-off function satisfying that

$$\chi(t) = 1, \text{ for } t \in [0, \varepsilon/4], \quad \chi(t) = 0, \text{ for } t \in [3\varepsilon/4, \infty)$$

for some small $\varepsilon > 0$. See Figure 3. Pull it back to $\nu(S)$ via

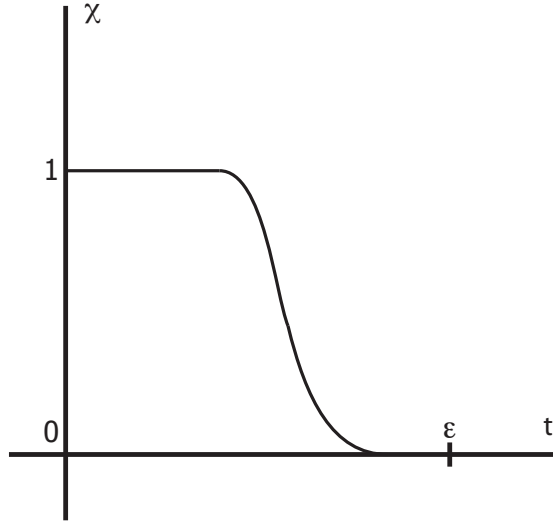


FIGURE 3. Cut-off function.

$$\begin{aligned} \chi_{\tilde{S}, \varepsilon} : \nu(S) &\longrightarrow [0, 1] \\ v &\longrightarrow \chi(\|v\|). \end{aligned}$$

Finally, define $\chi_{S, \varepsilon} = \chi_{\tilde{S}, \varepsilon} \circ \phi^{-1}$ on a neighborhood of S in M and extend it by zero to the whole manifold M . We have the following nowhere-vanishing

1-form

$$\alpha_1 = \chi_{S,\varepsilon} \cdot \widehat{\alpha}_1 + (1 - \chi_{S,\varepsilon}) \cdot \alpha.$$

Actually, we have the following linear family

$$\alpha_t = t\alpha_1 + (1 - t)\alpha$$

interpolating between α and α_1 . These forms are defined all over M (changes happen in a very small neighborhood of S). The family $\{\xi_t = \ker \alpha_t\}$ is C^1 -small provided that $\varepsilon > 0$ is small, since α and α_1 coincide over S . We also have

$$\omega_1 = \chi_{S,\varepsilon} \cdot d\alpha_1 + (1 - \chi_S) \cdot \omega,$$

defined over the whole manifold M . So, taking the constant ε small enough, we can make sure that the family $\{t\omega + (1 - t)\omega_1\}$ is a C^0 -small parametric family of symplectic forms over ξ_t . Notice that they are not necessarily closed. In fact, they fail to be closed in the support of $d\chi_S$ that can be chosen arbitrarily close to S . \square

Thus, if the base locus B and C of a quasi-contact pencil are contact we are able to produce a distribution which is contact near in a neighborhood of them. We now state the actual aim of this section, a result providing an almost contact pencil with the suitable properties to apply our argument:

Corollary 4.4. *Let (M, ξ, ω) be a quasi-contact closed 5-dimensional manifold and let (f, B, C) be a quasi-contact pencil. There exists a C^0 -small perturbation (M, ξ', ω') of the almost contact structure¹, for which (f, B, C) is still adapted, such that:*

- (i) *the perturbed almost contact structure is still quasi-contact away from arbitrarily small neighborhoods of C and B ,*
- (ii) *$\xi' = \ker \alpha'$ is contact in a neighborhood of the submanifolds C and B ; further $d\alpha' = \omega'$ in that neighborhood,*
- (iii) *the restriction of the distribution ξ' to the fibres of f has vanishing first Chern class.*

Proof. We start by constructing the adapted pencil provided by Corollary 3.5. Then, we can just apply Proposition 4.3 if the submanifolds C and B happen to be contact submanifolds of the quasi-contact structure. But this is obvious since C and B are transverse links and therefore contact submanifolds. Recall that if the perturbation is C^0 -small the quasi-contact pencil is still adapted to the new quasi-contact structure. In the same way, the first Chern class of the restriction of the distribution to the fibers do not change after a small perturbation of the distribution. \square

Note that the dimension of M plays a role in this Corollary since the submanifold B has codimension 4 and it is a link only if the dimension of

¹The quasi-contact condition is no longer satisfied. The new distribution is just an almost contact structure.

M is 5. This obstruction is not essential.

In this article we will be using the following:

Definition 4.5. *An almost contact pencil (f, C, B) adapted to an almost contact structure (M, ξ, ω) is called good, if the almost contact structure is contact in some sufficiently small neighborhoods of B and C .*

We have shown so far that any 5-dimensional almost contact pencil can be assumed to be good (Corollary 4.4).

5. CONTACT BLOW-UP.

We define an almost contact (resp. contact) fibration to be an almost contact (resp. contact) pencil with empty base points set B . We will mimic algebraic geometry in order to produce a fibration out of a pencil by blowing-up the base locus. The difficulty is that we need the notion of contact blow-up. The aim of this section is to provide such a concept.

We remark that the notion of contact blow-up was already introduced in M. Gromov's book *Partial differential relations* as a set of exercises. In this section we provide the necessary details. This notion is also related to the contact cuts introduced in [Ler]. We provide in this Section the definition for blow-up of transverse loops, the equivalent of blow-up of points in the symplectic setting. The general notion will be treated in a separate paper [CPP]. The main obstacle to provide a general definition is the lack of canonicity of the symplectic blow-up along symplectic submanifolds either in the surgery approach [McD] or in the reduction approach (Section 7.1 in [MS]).

However, we will precisely define the contact blow-up of transverse loops as a surgery operation (see Subsection 5.1). Let us first give the geometric intuition: it follows now an informal geometric interpretation of the surgery scheme developed later on. Recall that associated to a symplectic manifold (W, ω) such that $[\omega]$ lifts to an element of $H^2(W, \mathbb{Z})$, there is a canonical hermitian line bundle L with connection ∇_L such that its curvature is $-i\omega$. The associated unit circle bundle $\mathbb{S}(W)$ is called the Boothby-Wang manifold and it is well-known that the restriction of the connection to it provides a canonical contact structure [BW].

We start with a contact manifold (M, ξ) and a contact submanifold S such that $S = \mathbb{S}(W) \xrightarrow{\pi} W$ is itself a Boothby-Wang manifold. Assume also that the conformal symplectic normal bundle $\nu(S)$ is a pull-back of a conformal symplectic bundle $V \rightarrow W$, i.e. $\nu(S) = \pi^*V$. Gromov claims that in this setting there is a notion of contact blow-up. The main steps in the construction are:

- (i) Show using Gray's stability that there is a neighborhood U of $S = \mathbb{S}(W) \subset M$ that is contactomorphic to the Boothby–Wang manifold associated to $U' \subset V$ a small neighborhood of the zero section of V , with an adequate choice of symplectic structure over V .
- (ii) Taking advantage of the *conformal* symplectic structure of the normal bundle, rescale the construction in such a way that the radius of U' is at least greater than 1. This is performed by a squeezing of the neighborhood.
- (iii) Perform the symplectic blow-up \tilde{V} of the zero section inside V and recall that the cohomology class depends on the radius. Choose radius 1 to guarantee that the blow-up symplectic form is of integer class ([McD] provides a formula depending on the radius).
- (iv) Construct the Boothby–Wang manifold $\mathbb{S}(\tilde{V})$, we are able to do so because of the radius condition. A neighborhood of the exceptional divisor will be $\mathbb{S}(\tilde{U}') \subset \mathbb{S}(\tilde{V})$.
- (v) Replace the neighborhood U of S by $\mathbb{S}(\tilde{U}')$. For that it is needed to check that $U \setminus S$ is contactomorphic to $\tilde{U}' \setminus \tilde{W}$.

The necessary details will appear in [CPP]. The example that we will carefully develop is that of a transverse loop S in a contact manifold of dimension $2n + 1$, greater or equal to 5. The loop is the Boothby–Wang manifold of a point $W = \{pt\}$. The normal bundle is trivial and therefore satisfies the pull-back condition. All the previous discussion just amounts to a surgery that replaces the loop S (blow-up locus) by the Boothby–Wang manifold associated to \mathbb{CP}^{n-1} , that is the standard contact sphere \mathbb{S}^{2n-1} . This sphere will be called the exceptional divisor E . So, the contact blow-up of M along S , denoted by \tilde{M} , is a surgery operation that replaces a neighborhood of the loop diffeomorphic to $\mathbb{S}^1 \times B^{2n}$ by $B^2 \times \mathbb{S}^{2n-1}$ identifying the common boundaries $\mathbb{S}^1 \times \mathbb{S}^{2n-1}$. Recall that topologically this is just an index 1 surgery on the manifold.

Unlike in the symplectic (or algebraic) case, there is no natural projection map from \tilde{M} to M . In fact, there is not even a natural projection from $E = \mathbb{S}(W)$ to $S = \mathbb{S}(V)$, for instance in the 5 dimensional case we have that $E = \mathbb{S}^3$ and $S = \mathbb{S}^1$. What is still true is that $M \setminus L$ is contactomorphic to $\tilde{M} \setminus E$.

Observe that there is a choice of symplectic structure ω_W on the submanifold W . Fix a positive integer $k \in \mathbb{Z}^+$. What happens if we choose $(W, k\omega_W)$? Then the associated Boothby–Wang manifold changes, even topologically. For instance in the 5-dimensional case we obtain that the sphere bundles over \mathbb{CP}^1 associated to the different symplectic forms $m\omega_{\mathbb{CP}^1}$ are the lens spaces $L(k, 1)$ with their standard contact forms inherited as quotients of $\mathbb{S}^3 = L(1, 1)$. In other words, the choice of collapsing radius –

which determines $k \in \mathbb{Z}^+$ – changes the topology of the exceptional divisor. This just means that the contact blow-up is even less canonical than the symplectic blow-up. This is hardly a surprise at this point of the discussion.

5.1. Contact blow-up surgery. We now define precisely the contact blow-up along loops in the following:

Theorem 5.1. *Let (M^{2n+1}, ξ) be a contact manifold. Let $S \subset M$ be a smooth transverse loop in M . There exists a manifold \widetilde{M} – called the contact blow-up of M along S – satisfying the following conditions:*

- *There exists a contact structure $\tilde{\xi}$ on \widetilde{M} , i.e. the blow-up of M along S is a contact manifold.*
- *There exists a codimension 2 contact submanifold E inside \widetilde{M} with trivial normal bundle. Moreover the manifold E is contactomorphic to the sphere \mathbb{S}^{2n-1} , with its standard contact structure.*
- *The manifolds $(M \setminus S, \xi)$ and $(\widetilde{M} \setminus E, \tilde{\xi})$ are contactomorphic.*

Note that this definition coincides with the previous informal definition of contact blow-up for the case of a transverse loop. Indeed, if $k = 1$ the Boothby–Wang manifold associated to \mathbb{CP}^{n-1} is the standard contact sphere \mathbb{S}^{2n-1} .

Let $\mathbb{S}^{2n-1} \xrightarrow{e} \mathbb{R}^{2n}$ be the standard inclusion of the unit sphere in \mathbb{R}^{2n} . Fix the standard Liouville form of \mathbb{R}^{2n} , $\lambda = \sum_j x_j dy_j - y_j dx_j$. We define the standard contact form on \mathbb{S}^{2n-1} as $\alpha_{std} = e^* \lambda$. We first need to understand the variation of the contact structure under the Hopf action:

Lemma 5.2. *Let $\mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1}$ have coordinates $(\theta, r, w_1, \dots, w_n)$, with $(w_1, \dots, w_n) \in \mathbb{S}^{2n-1} \subset \mathbb{C}^n$. Let ξ_0 be the contact structure defined as $\xi_0 = \ker\{\eta_0\}$, where the form η_0 is defined as*

$$(3) \quad \eta_0 = d\theta - r^2 \alpha_{std}.$$

Define the diffeomorphism

$$\begin{aligned} \phi_1 : \mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1} &\longrightarrow \mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1} \\ (\theta, r, w_1, \dots, w_n) &\longrightarrow (\theta, r, e^{i\theta} w_1, \dots, e^{i\theta} w_n). \end{aligned}$$

Then $\phi_1^ \xi_0 = \ker\{\alpha_{std} + \frac{r^2-1}{r^2} d\theta\}$.*

The proof of the Lemma is a straightforward computation.

Secondly, we need a *squeezing* result to ensure the radius condition. Let $\mathbb{S}^1 \times B^{2n}(R) \subset \mathbb{S}^1 \times \mathbb{R}^{2n}$ be the Boothby–Wang manifold of the radius R ball and use *generalized polar* coordinates $(\theta, r, w_1, \dots, w_n) \in \mathbb{S}^1 \times [0, R) \times \mathbb{S}^{2n-1}$ in it. Then we have the following *squeezing* property:

Lemma 5.3. (*Proposition 1.24 in [EKP]*) *For any integer $k > 0$ and radius $0 < R_0$, the following map is a contactomorphism*

$$\begin{aligned} \psi_k : \mathbb{S}^1 \times B^{2n}(R_0) &\longrightarrow \mathbb{S}^1 \times B^{2n}\left(\frac{R_0}{\sqrt{1+k^2 R_0^2}}\right) \\ (\theta, r, w_1, \dots, w_n) &\longrightarrow \left(\theta, \frac{r}{\sqrt{1+k^2 r^2}}, e^{ik\theta} w_1, \dots, e^{ik\theta} w_n\right), \end{aligned}$$

and it restricts to the identity at $\mathbb{S}^1 \times \{0\}$.

This Lemma says that we can contract the radius of the tubular neighborhood by a contact transformation preserving the central fibre.

Proof of Theorem 5.1. By Gray's stability there is a neighborhood U of S that is contactomorphic to the manifold $\mathbb{S}^1 \times B^{2n}(\varepsilon)$ with the contact structure defined as the kernel of the form (3). We denote the map as $\Phi : \mathbb{S}^1 \times B^{2n}(\varepsilon) \longrightarrow U$. Choose an integer k satisfying that

$$\frac{2}{\sqrt{1+4k^2}} < \varepsilon.$$

This implies that, the map $\psi_k : \mathbb{S}^1 \times B^{2n}(2) \longrightarrow \mathbb{S}^1 \times B^{2n}(\varepsilon)$, defined in Lemma 5.3, is a contact embedding. So we obtain a contact embedding from a radius 2 neighborhood of the loop into the manifold, namely

$$\Phi_k = (\Phi \circ \psi_k) : \mathbb{S}^1 \times B^{2n}(2) \longrightarrow U \subset M.$$

We use the map $\phi_1 : \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1} \longrightarrow \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1}$ provided by Lemma 5.2. Denote $V = \Phi_k(\mathbb{S}^1 \times B^{2n}(3/2))$. We have that $\rho = \Phi_k \circ \phi_1 : \mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^{2n-1} \longrightarrow U \setminus V \subset M$ satisfies that

$$\rho^* \xi = \ker \left\{ \alpha_{std} + \frac{r^2 - 1}{r^2} d\theta \right\}.$$

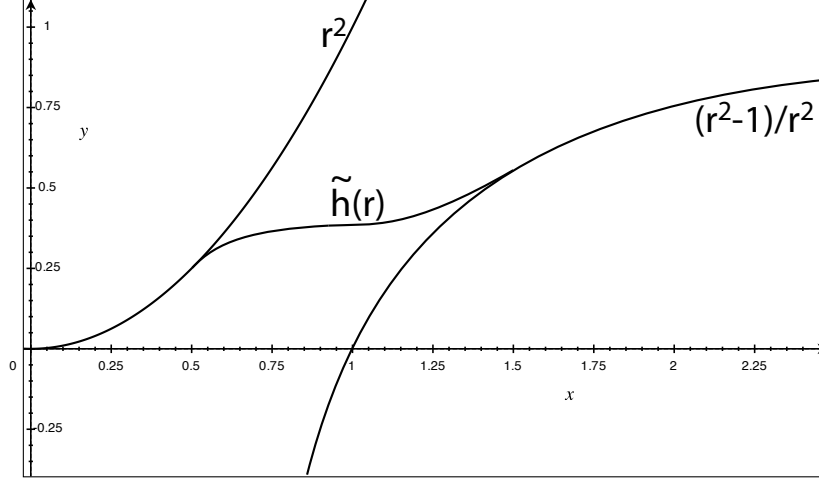
Note that the function

$$\begin{aligned} h : (3/2, 2) &\longrightarrow \mathbb{R} \\ r &\longrightarrow h(r) = \frac{r^2 - 1}{r^2} \end{aligned}$$

satisfies $h(r) > 5/9$. Therefore it is possible to extend it to a smooth function $\tilde{h} : [0, 2) \longrightarrow \mathbb{R}$ satisfying the following conditions (see Figure 4):

- $\tilde{h}(r) = r^2$, for $r \in [0, 1/2]$,
- $\tilde{h}(r) = h(r)$, for $r > 3/2$,
- $\tilde{h}(r)' > 0$ for $r \in [1/2, 3/2]$.

Therefore we have that $\tilde{\eta} = \alpha_{std} + \tilde{h}(r)d\theta$ is a contact form over $\mathbb{S}^1 \times [0, 2) \times \mathbb{S}^{2n-1} \simeq B^2(2) \times \mathbb{S}^{2n-1}$. We can glue the manifold $(M \setminus V, \xi)$ and $(B^2(2) \times \mathbb{S}^{2n-1}, \ker\{\tilde{\eta}\})$ with the gluing map ρ to define a new contact manifold that we denote as $(\tilde{M}, \tilde{\xi})$, the contact blow-up of M . \square

FIGURE 4. The function \tilde{h} .

The choice of a different radius in the symplectic blow-up amounts to the use of the map

$$\begin{aligned} \phi_k : \mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1} &\longrightarrow \mathbb{S}^1 \times (0, 2) \times \mathbb{S}^{2n-1} \\ (\theta, r, w_1, \dots, w_n) &\longrightarrow (\theta, r, e^{ik\theta} w_1, \dots, e^{ik\theta} w_n), \end{aligned}$$

which satisfies $\phi_k^* \xi_0 = \ker\{\alpha_{std} + \frac{r^{2k-1}}{r^2} d\theta\}$ and leads to a different compactification by a lens space $L(k, 1)$. This will be developed in [CPP].

5.2. Compatibility with a contact pencil. There are several choices in the previous construction. One of them is that of the map $\Phi : \mathbb{S}^1 \times B^{2n}(\varepsilon) \longrightarrow U$. This amounts to a choice of framing of the trivial normal bundle to S . If we have a contact pencil (f, C, B) in a five dimensional manifold M and S is one of the connected components of B we can fix Φ by using the approximately holomorphic charts to satisfy that the map

$$f \circ \Phi : \mathbb{S}^1 \times (B^4(\varepsilon) \setminus \{0\}) \longrightarrow \mathbb{CP}^1$$

is precisely $(f \circ \Phi)(\theta, w_1, w_2) = [w_1 : w_2]$. Therefore the compactified fibres are of the form $\mathbb{S}^1 \times L$, for any complex line $L \in \mathbb{C}^2$.

We want to understand the compactification of fibres in the blow-up. We first restrict ourselves to the transition region $\mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^3 \subset \mathbb{S}^1 \times \mathbb{C}^2$. The gluing map is $\rho = \phi_1 \circ \Phi_k = \phi_1 \circ \psi_k \circ \Phi$, in order to understand the fibres of the blown-up pencil, we must study the map $\tilde{f} = f \circ \Phi \circ \psi_k \circ \phi_1$. It is simple to check that $\tilde{f}(\theta, r, w_1, w_2) = (f \circ \Phi)(\theta, rw, rw_2) = [w_1 : w_2]$, since $\psi_k \circ \phi_1$ acts as complex scalar multiplication in the transition area.

Notice that the domain of definition of \tilde{f} is $\mathbb{S}^1 \times (3/2, 2) \times \mathbb{S}^3$, and it is invariant with respect to the coordinates $(\theta, r) \in \mathbb{S}^1 \times (3/2, 2)$. Hence, the map \tilde{f} extends trivially to the model $(B^2(2) \times \mathbb{S}^3, \ker \tilde{\eta})$. In particular, the extension of \tilde{f} restricted to the exceptional divisor $\{0\} \times \mathbb{S}^3$ consists of the Hopf fibration.

The fibres of the blown-up fibration are contact submanifolds. Indeed, the fibres of \tilde{f} restricted to $(B^2(2) \times \mathbb{S}^3, \ker \tilde{\eta})$ are $B^2(2) \times \mathbb{S}^1$, the \mathbb{S}^1 being a Hopf fibre, and these submanifolds are certainly contact.

This previous discussion can be summarized in the following

Lemma 5.4. *Let (f, B, C) be a contact pencil for the 5-manifold (M, ξ) . With the previous choices its contact blow-up $(\tilde{M}, \tilde{\xi})$ has a contact fibration (\hat{f}, \hat{C}) that coincides with (f, C) away from the blow-up locus. The fibres of this fibration are contactomorphic to the compactified fibres of the initial pencil. Moreover, the restriction of the map \hat{f} to each of the exceptional divisors is given by the Hopf fibration.*

Remark 5.5. *The previous lemma is of local nature and therefore it applies with the obvious changes to almost contact pencils that are “good”. In that case, we obtain an almost contact fibration that is contact in neighborhoods of the exceptional divisors and of the critical points. This will be called a good almost contact blown-up fibration.*

5.3. Contact blow-down in 5 dimensions. The argument will be developed using a fibration obtained by blowing up the base locus of the almost contact pencil, in order to recover the initial manifold we need the converse procedure:

Lemma 5.6. *Let M be a contact manifold and let E be a 3-dimensional contact submanifold with trivial normal bundle that happens to be the standard contact 3-sphere, then we can replace the sphere by a knot K and find a contact structure in the new manifold \overline{M} such that the knot becomes transverse. Moreover the manifolds $M \setminus E$ and $\overline{M} \setminus K$ are contactomorphic.*

Proof. The same construction that in the blow-up case holds. We just use ϕ_1^{-1} to glue the other way around. This time even $k = 1$ can be selected. \square

Remark that the diffeomorphism type of the blown-up manifold depends on the choice of $k > 0$. Even choices of k not being diffeomorphic to odd choices. When we apply the blow-up and blow-down process in this article for a given transverse loop, we choose the same k for both processes. That, at least, preserves the homotopy type of the contact structure (understood as an almost contact structure).

6. VERTICAL CONTACT PENCILS.

In the previous sections we have obtained a contact structure in a neighborhood of the base locus and the critical curves. In order to obtain a contact structure in the manifold we use the splitting induced by the existence of a fibration; we shall first achieve the contact condition in the fibre direction, which will be referred as the *vertical* direction, and then proceed with the *horizontal* direction, i.e. the one induced by the base, normal to the fibres. The former will be the aim of this section, whereas the latter will be dealt with in the next sections. In other words, we first want to homotope the given almost contact distribution supported by an almost contact fibration to a distribution admitting an almost contact fibration with non-singular fibres being contact. Let us define such a concept in the following

Definition 6.1. *An almost contact structure is called vertical with respect to an almost contact fibration if the fibres are contact submanifolds away from the critical points.*

In this section, we assume that M is a 5-dimensional closed orientable almost contact manifold. The theory developed so far does not require any condition on the dimension of the manifold, nevertheless it will be essential for the argument to apply that the fibres of the almost contact fibration are 3-dimensional manifolds. Compared to higher dimensions the contact topology in dimension 3 is better understood. In order to make the fibration vertical we will need Eliashberg's classification result of overtwisted contact structures [El]. The main result of this section can be stated as:

Theorem 6.2. *Let (f, C) be a good almost contact blown-up fibration with non-empty set of exceptional divisors E_j . Then there exists a homotopic deformation of the almost contact structure such that the almost contact structure becomes vertical for the fibration. Moreover, the deformation is supported away from C and E_j .*

The proof of the theorem will rely on the existence of an overtwisted disk in each fibre, such structure allows more flexibility in handling families of distributions. The precise meaning of this will now be established.

6.1. 3-dimensional Overtwisted Structures. Our setup provides a fibration with a distribution on each fibre. This can be locally reduced to Eliashberg's setup. Indeed, given an almost contact fibration $M \rightarrow \mathbb{CP}^1$, let F_z denote the fibre over $z \in \mathbb{CP}^1$ and denote by (ξ_z, ω_z) the induced almost contact structure on F_z . Then the family (F_z, ξ_z) can locally be viewed as a 2-parametric family of 2-distributions on a fixed fibre.

The crucial step in the proof of Theorem 6.2 will be a relative version of the following result due to Eliashberg:

Theorem 6.3 (Theorem 3.1.1 in [El]). *Let M be a compact closed 3-manifold and let G be a closed subset such that $M \setminus G$ is connected. Let*

K be a compact space and L a closed subspace of K . Let $\{\xi_t\}_{t \in K}$ be a family of cooriented 2-plane distributions on M which are contact everywhere for $t \in L$ and are contact near G for $t \in K$. Suppose there exists an embedded 2-disk $\mathcal{D} \subset M \setminus G$ such that ξ_t is contact near \mathcal{D} and (\mathcal{D}, ξ_t) is equivalent to the standard overtwisted disk for all $t \in K$. Then there exists a family $\{\xi'_t\}_{t \in K}$ of contact structures of M such that ξ'_t coincides with ξ_t near G for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. Moreover ξ'_t can be connected with ξ_t by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.

We will use a version allowing the case of a 3-manifold with boundary:

Corollary 6.4. *Let M be a compact 3-manifold with boundary ∂M and let G be a closed subset of M such that $M \setminus G$ is connected and $\partial M \subset G$. Let K be a compact space and L a closed subspace of K . Let $\{\xi_t\}_{t \in K}$ be a family of cooriented 2-plane distributions on M which are contact everywhere for $t \in L$ and are contact near G for $t \in K$. Suppose there exists an embedded 2-disk $\mathcal{D} \subset M \setminus G$ such that ξ_t is contact near \mathcal{D} and (\mathcal{D}, ξ_t) is equivalent to the standard overtwisted disk for all $t \in K$. Then there exists a family $\{\xi'_t\}_{t \in K}$ of contact structures of M such that ξ'_t coincides with ξ_t near G for $t \in K$ and coincides with ξ_t everywhere for $t \in L$. Moreover ξ'_t can be connected with ξ_t by a homotopy through families of distributions that is fixed in $(G \times K) \cup (M \times L)$.*

Outline. The proof for the closed case uses a suitable triangulation P of the 3-manifold having a subtriangulation Q containing G , for which the distributions are already contact structures. Then Eliashberg's argument is of local nature, working with neighborhoods of the 0, 1, 2 and 3-skeleton of $P \setminus Q$ and assuring that no changes are made in a neighborhood of Q . Hence, the method is still valid since P and Q do exist in the case of a manifold with boundary, and only Q contains the boundary. \square

Recall that we will locally treat an almost contact fibration as a 2-parametric family of distributions over a fixed fibre, thus we may use a disk as a parameter space and the central fibre as the fixed manifold. It will be useful to be able to obtain a continuous family of distributions such that the distributions in a neighborhood of the central fibre become contact structures while the distributions near the boundary are fixed. Such a family is provided in the following

Corollary 6.5. *Under the same hypothesis and notation of Corollary 6.4, assume that K is a topological ball, so $SS = \partial K$ is a sphere. Let*

$$\tilde{K} = K \cup_{\partial} (SS \times [0, 1]) \quad \text{and} \quad \tilde{L} = L \cup_{\partial} (\partial L \times [0, 1])$$

where we identify $\partial K = SS \cong SS \times \{0\}$ and $\partial L \cong \partial L \times \{0\}$. Let the family of distributions be defined for $\{\xi_t\}_{t \in \tilde{K}}$, the distributions being contact near G and \mathcal{D} . Then there exists a deformation $\{\tilde{\xi}_t\}_{t \in \tilde{K}}$ such that:

- Satisfies the properties obtained in Corollary 6.4 when the family of parameters is restricted to K . In particular, for any $t \in K$ the distributions ξ_t and $\tilde{\xi}_t$ coincide on an open neighborhood of $G \cup \mathcal{D}$.
- The distributions ξ_t and $\tilde{\xi}_t$ coincide over M for $t \in \partial\tilde{K} \cong SS \times \{1\}$. Further, $\tilde{\xi}_t$ can be connected with ξ_t by a homotopy through families of distributions fixed in $(G \times \tilde{K}) \cup M \times (\tilde{L} \cup \partial\tilde{K})$.

Proof. Corollary 6.4 provides the family $\tilde{\xi}_t$ for $t \in K$, it is left to extend the parameter over \tilde{K} . Since both families must coincide at $SS \times \{1\}$, we use the structures ξ_t in the annulus $SS \times [1/2, 1]$ and start deforming to $\tilde{\xi}_t$ at $SS \times \{1/2\}$ until we reach $\partial K \cong SS \times \{0\}$, where it glues with $\tilde{\xi}_t$.

In more precise terms, let us denote the family ξ_t restricted to $SS \times [0, 1]$ by $\{\eta_p^s\}_{p \in SS}^{s \in [0,1]}$, i.e. $\xi_t = \xi_{(p,s)} = \eta_p^s$. Let $\{\xi_t^s\}_{t \in SS}^{s \in [0,1]}$ be the deformation of $\xi_{(t,0)} = \eta_t^0 = \xi_t^0$ to $\tilde{\xi}_{(t,0)} = \xi_t^1$, restricted to $SS \times \{0\} \simeq \partial K$. Define the continuous deformation

$$\{\nu_t^s\}_{t \in SS}^{s \in [0,1]} = \begin{cases} \eta_t^{1-2s}, & s \in [0, 1/2], \\ \xi_t^{2s-1}, & s \in [1/2, 1]. \end{cases}$$

Finally, the following deformation extends $\{\xi_t^s\}$ to the domain $SS \times [0, 1]$

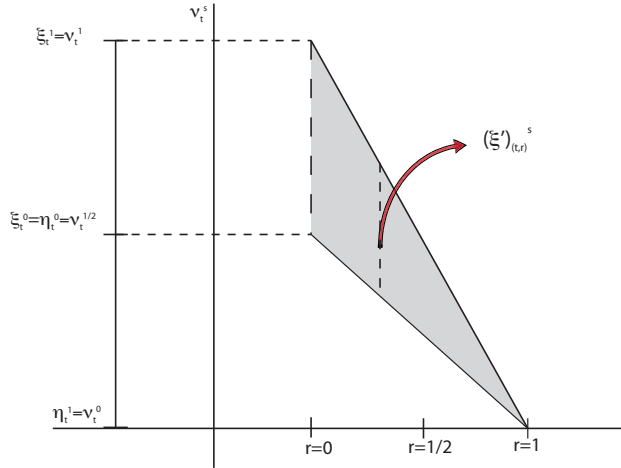


FIGURE 5. Deformation in the boundary of the ball. $t \in SS$ is fixed.

and satisfies all the required properties:

$$\{(\tilde{\xi})_{(t,r)}^s\}_{(t,r) \in SS \times [0,1]}^{s \in [0,1]} = \{\nu_t^q\}, \quad q = \frac{1}{2}(1+s)(1-r).$$

See Figure 4 for a visual realization. \square

Remark 6.6. *Notice that the assumption that K is a topological ball is not necessary. However, since we will apply Corollary 6.5 when K is a topological ball, we consider more appropriate to state it in this particular case.*

Now, we need to produce an overtwisted disk over each fibre in order to apply Corollary 6.4. We need at least one over each fibre and we will need to behave them *continuously*. The next step is thus to find such a family of disks.

6.2. Families of overtwisted disks. There are two basic issues to be treated: the location of the disks and their overtwistedness. The second is guaranteed since once we know where the disk can be placed we will produce an overtwisted disk using a Lutz twist.

In order to decide the location of the disk in each fibre we first need to find a section of the almost contact fibration. The base manifold is \mathbb{CP}^1 , these fibrations are well-understood in algebraic geometric terms and also appear in the topology of 4-manifolds: for instance, studying Hirzebruch surfaces or more generally elliptic fibrations. Given a fibration E over \mathbb{CP}^1 one can produce a new fibration with a section by considering $\mathbb{P}(E \oplus \mathbb{C})$, the section *at infinity* is a global section, or using a blow-up, e.g. $\mathbb{CP}^2 \# 9\mathbb{CP}^2$ fibering over \mathbb{CP}^1 has 9 preferred global sections, namely mapping the base \mathbb{CP}^1 to each exceptional divisor \mathbb{CP}^1 . Hence, in the symplectic Lefschetz pencil case having a non-empty base locus to blow-up will at the same time produce a fibration and a section since the exceptional divisor does indeed intersect all the fibres once. To apply these ideas to our setting we have to modify a bit the strategy since we already achieved the contact condition in a neighborhood of the critical curves and the exceptional divisors. Note however that the base loci of the almost contact pencils constructed are non-empty after Corollary 3.5.

Let (f, C) be the almost contact blown-up fibration obtained as in the previous sections. Denote by $U(C), U(E_i)$ open neighborhoods of the critical curves C and the exceptional spheres E_i of an almost contact fibration obtained after blowing up along B . Let $U(f) = U(C) \cup U(E_i)$ be the union of these open neighborhoods, so in the complementary of $U(f)$ the almost contact fibration associated to the pencil becomes a submersion. Instead of finding a global section mapping away from $U(f)$, we shall construct two *disjoint* sections that will provide at least one overtwisted disk in each fibre. The global situation we will achieve is described as follows:

Proposition 6.7. *Let (M, f) be an almost contact fibration with a set of critical curves C and exceptional spheres E_i . There exists a deformation $(F_z, \tilde{\xi}_z)_{z \in \mathbb{CP}^1}$ of the family $(F_z, \xi_z)_{z \in \mathbb{CP}^1}$ fixed at the intersection of the set $U(f)$ with each F_z satisfying:*

- (i) *There exist two open disks $\mathcal{B}_0, \mathcal{B}_\infty \subset \mathbb{CP}^1$, containing 0 and ∞ respectively such that the intersection $\mathcal{B}_0 \cap \mathcal{B}_\infty$ is an open annulus and the curves $\partial \mathcal{B}_i$ are disjoint from the set of curves $f(C)$.*
- (ii) *There exists two disjoint families of embedded 2-disks $\mathcal{D}_z^i \subset F_z$, with $z \in \mathcal{B}_i$, for $i = 0, 1$, not intersecting $U(f)$. Further, the structure $\tilde{\xi}_z$ is contact in a neighborhood of such families and $(\mathcal{D}_z^i, \tilde{\xi}_z)$ are equivalent to standard overtwisted disks.*

Remark 6.8. *The complement of $\mathcal{B}_0 \cap \mathcal{B}_\infty$ consists of two disjoint disks. It will be clear, out of the proof, that the diameter of those disks can be chosen as small as desired. This is not needed for the proof of the results of this Section, but it will be needed later on.*

The fact that $\tilde{\xi}_z$ equals ξ_z in the intersection of the set $U(f)$ with F_z ensures that no perturbation is performed near the critical curves nor the exceptional spheres. This is mainly a global statement, involving the whole of the fibres. In order to prove the result we have to understand the nature of the following *reference model*: a tubular neighborhood of an exceptional divisor.

Let K be a knot belonging to the base locus B of the almost contact pencil. After the blow-up procedure it is replaced by an exceptional contact divisor $E \cong \mathbb{S}^3$ with the standard contact structure. Recall from Section 5 that the restriction of the fibration \tilde{f} to E is the Hopf fibration. Since the distribution ξ is locally a contact structure $\{\xi_e = \ker \alpha\}$, the tubular neighborhood theorem provides a chart

$$(4) \quad \Psi : U \longrightarrow \mathbb{S}^3 \times B^2, \quad \Psi^* \xi_{st} = \xi$$

where $\xi_{st} = \ker\{\alpha_{\mathbb{S}^3} + r^2 d\theta\}$. The fibres of the induced map \tilde{f}_U defined as

$$\begin{array}{ccc} \mathbb{S}^3 \times B^2 & \xrightarrow{\Psi^{-1}} & U \\ & \searrow \tilde{f}_U & \downarrow \tilde{f} \\ & & \mathbb{CP}^1 \end{array}$$

correspond to tubular neighborhoods of the Hopf fibres, that is to say $(\mathbb{S}^1 \times B^2, \xi_v = \ker(d\beta + r^2 d\theta))$ for each $z \in \mathbb{CP}^1$. Note that the variable $\beta \in \mathbb{S}^1$ parametrizing each Hopf fibre is not global since the fibration is not trivial, however the differential $d\beta$ is globally well-defined since it is dual to the vector field generating the associated \mathbb{S}^1 action. Using an almost contact

connection² the standard contact structure in $\mathbb{S}^3 \times B^2$ can be expressed as the direct sum of distributions

$$(5) \quad \xi_{st} = \xi_v \oplus H,$$

where ξ_v is the standard contact structure in $\mathbb{S}^1 \times B^2$, *the vertical direction*, and H is the contact³ connection associated to the fibration of $\mathbb{S}^3 \times B^2$ over \mathbb{CP}^1 . Topologically, the 4-distribution ξ_{st} is expressed as a direct sum of two distributions of 2-planes. Since the 2-form ω providing the almost contact structure is given and so is ξ , we may interpret $(\mathbb{S}^3 \times B^2, \xi_v)$ as a non-trivial family of contact structures parametrized by the base, $\{\xi_q = \xi_v\}$, $q \in \mathbb{CP}^1$. So far we understand the topology and contact structure of the *reference* model seen as a fibration: indeed, in a neighborhood of the exceptional divisor it is precisely a piece of the blown-up fibration and the knots are the intersection of the fibres of the almost contact pencil with an exceptional sphere.

We state the local version of Proposition 6.7 which shall be proven using the model mentioned above. From this local result, it will be straightforward to obtain the suitable global families of disks:

Proposition 6.9. *There exists a perturbation of the contact structures $(\mathbb{S}^3 \times B^2, \xi_v)$ relative to a neighborhood of the boundary such that the perturbed structures are contact and with respect to them:*

- (i) *There exists two continuous families of overtwisted disks $\{\mathcal{D}_q^0\}_{q \in \mathcal{B}^0}$ and $\{\mathcal{D}_q^\infty\}_{q \in \mathcal{B}^\infty}$. Over each point $q \in \mathcal{B}_0 \setminus \partial \mathcal{B}_0$ (resp. $q \in \mathcal{B}_\infty \setminus \partial \mathcal{B}_\infty$) there is an overtwisted disk \mathcal{D}_q^0 (resp. \mathcal{D}_q^∞).*
- (ii) *At every $q \in \mathcal{B}_0 \cap \mathcal{B}_\infty$, the disks \mathcal{D}_q^0 and \mathcal{D}_q^∞ are disjoint.*

We understand $\mathcal{B}_0 \setminus \partial \mathcal{B}_0, \mathcal{B}_\infty \setminus \partial \mathcal{B}_\infty$ as neighborhoods of the upper and lower semi-spheres. Let us prove the result:

Proof. Let $h : \mathbb{S}^3 \rightarrow \mathbb{CP}^1$ be the Hopf fibration, extend the fibration to $h : \mathbb{S}^3 \times B^2 \rightarrow \mathbb{CP}^1$ by projection onto the first factor. Let $\mathcal{B}_0, \mathcal{B}_\infty$ be two disks containing $0, \infty \in \mathbb{CP}^1$, e.g. the complements of a tubular neighborhood of ∞ and 0 . As explained before, the idea is to use the exceptional divisor to create a couple of sections. On the one hand, the exceptional divisor has a contact structure and we would rather not perturb around it, and on the other hand the exceptional divisor is no longer \mathbb{CP}^1 but \mathbb{S}^3 . We use two copies of the exceptional divisor away from $\mathbb{S}^3 \times \{0\} \subset \mathbb{S}^3 \times B^2$ and we trivialize the base \mathbb{CP}^1 with the two disks $\mathcal{B}_0, \mathcal{B}_\infty$.

²If (f, C) is an almost contact fibration for (M, ξ, ω) , an almost contact connection H_ξ is defined at a point $p \in M \setminus C$ as the distribution ω_p -symplectic orthogonal to the distribution of the vertical fibre $\xi_p \cap \ker df_p$ inside ξ_p .

³An almost contact connection is called a contact connection if the fibration and ξ are, with the compatibility condition: $\exists \alpha$ a contact form such that $\omega = d\alpha$ and $\xi = \ker \alpha$.

Let $q_0 = (1/2, 0)$, $q_\infty = (0, 1/2) \in B^2$ be two fixed points and denote the two corresponding 3-spheres

$$\mathbb{S}_0^3 = \mathbb{S}^3 \times \{q_0\}, \quad \mathbb{S}_\infty^3 = \mathbb{S}^3 \times \{q_\infty\}.$$

The fibre of the restriction of the fibration $(\mathbb{S}^3 \times B^2, \xi_v) \rightarrow \mathbb{CP}^1$ to the submanifold \mathbb{S}_0^3 (resp. \mathbb{S}_∞^3) is a transverse knot K_0^p (resp. K_∞^p). We will insert the family of overtwisted disks.

Applying a full Lutz twist (see [Lu, Ge]) in a small neighborhood of each of those knots $K_0^p \in h^{-1}(p)$ produces a 3-dimensional Lutz twist on each fibre. This family of overtwisted disks is parametrized as $\{\mathcal{D}_t^0\}_{t \in \mathbb{S}_0^3}$, thus we obtain a \mathbb{S}^1 -family of overtwisted disks at each fibre. Perform the same procedure for the family of knots $K_\infty^p \in h^{-1}(p)$ to obtain another family of disks $\{\mathcal{D}_t^\infty\}_{t \in \mathbb{S}_\infty^3}$. The two families of disks can indeed be disjoint by letting the radius in which we perform Lutz twists be small enough. This construction provides the perturbed family of contact structures, coinciding with the previous distribution near the boundary. Let us remark that the support of the pair of Lutz twists also does not intersect the exceptional divisor (see figure 6).

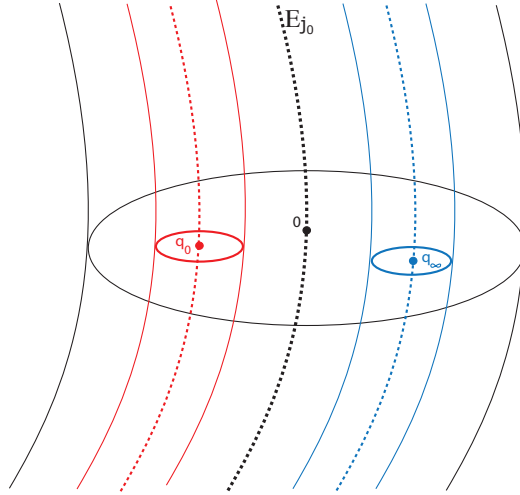


FIGURE 6. The neighborhood of the exceptional divisor intersected with the fibre. The red cylinder is the support of the Lutz twist around $\mathbb{S}^1 \times \{q_0\}$, the blue one around $\mathbb{S}^1 \times \{q_\infty\}$.

We need the base to be the parameter space, instead of a 3-sphere: restricted to \mathcal{B}_0 or \mathcal{B}_∞ the Hopf fibration becomes trivial and therefore there exist two sections $s_0 : \mathcal{B}_0 \rightarrow \mathbb{S}^3 \cong \mathbb{S}_0^3$ and $s_\infty : \mathcal{B}_\infty \rightarrow \mathbb{S}^3 \cong \mathbb{S}_\infty^3$. The required families are defined as

$$\{\mathcal{D}_q^0\} = \{\mathcal{D}_{s_0(q)}^0\}, q \in \mathcal{B}_0,$$

$$\{\mathcal{D}_q^\infty\} = \{\mathcal{D}_{s_\infty(q)}^\infty\}, q \in \mathcal{B}_\infty.$$

Note that the two families of overtwisted disks are disjoint since the two families of Lutz twists were. Further, there exists a small neighborhood of the exceptional divisor $\mathbb{S}^3 \times \{0\}$ where no deformation is performed. \square

Once the *reference* model is constructed it becomes an exercise to prove that the global setup can also be achieved:

Proof of Proposition 6.7. Apply the previous Proposition 6.9 to a neighborhood of one exceptional sphere E_{j_0} . The families of overtwisted disks do not meet C or any E_j ; indeed, the two families are arbitrarily close to the chosen exceptional divisor and the exceptional divisors are pairwise disjoint and none of them intersect the critical curves C . Thus the families are located away from $U(f)$ (slightly shrinking the neighborhood $U(E_{j_0})$). \square

We have thus obtained the families of overtwisted disks required to apply Eliashberg's theorem. The deformation procedure will be carried out using a suitable decomposition of the base \mathbb{CP}^1 ensuring the *vertical* contact condition at each piece. In what follows we construct such a decomposition and the proof of the main theorem of the section will be established right after.

6.3. Adapted families. Let (f, C) be a fixed almost contact fibration. A finite set of oriented immersed connected curves T in \mathbb{CP}^1 will be called an adapted family for \mathbb{CP}^1 if it satisfies the following properties:

- The image of the set of critical values $f(C)$ is part of T . Let C_i denote the image of each of the components of C .
- Given any element $c \in T$, there exists another element of $c' \in T$ having a non-empty intersection⁴ with c . Any two elements of T intersect transversally.

Let $|T| \subset \mathbb{CP}^1$ be the underlying set of points of the elements of T . The elements of an adapted family T that are not of the form C_i are denoted by F_j and will be called *fake* components.

The existence of a fine enough adapted family is assured since *fake* curves are allowed to be inserted, in other words:

Lemma 6.10. *There exists a sequence of adapted families T^n satisfying that the diameter of the set $\mathbb{CP}^1 \setminus |T|$ is $O(1/n)$.*

⁴In case c has a self-intersection, then $c' = c$ is allowed.

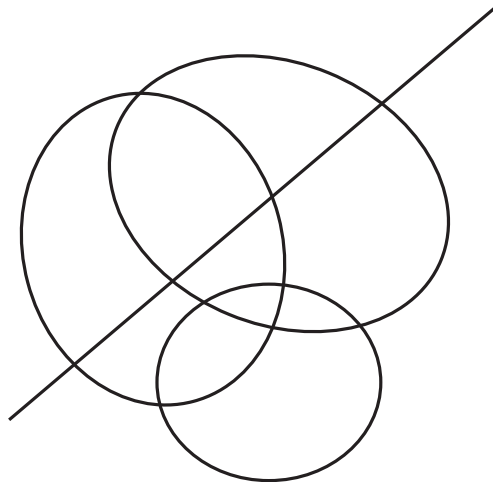


FIGURE 7. The figure here shows a part of an adapted family. The associated subdivision consists of certain polygons with their boundaries a union of parts of various elements in the family.

Proof. Consider the family $T^0 = \bigcup C_i$ and for each n add as many *fake* components as needed so that then new family remains adapted and it also satisfies the required properties. \square

To an adapted family there is an associated *triangulation*; more accurately, a cell decomposition of \mathbb{CP}^1 since the polygons, or 2-cells, are not necessarily triangles. See Figure 7. We shall first perturb in a neighborhood of each vertex relative to the boundary, proceed with a neighborhood of the 1-cells and finally obtain the *vertical* contact condition in the 2-cells. To be precise in the description of the procedure, we set some notation.

Let $L_j \in T$ be a curve, $U(L_j)$ be a tubular neighborhood and denote $\partial U(L_j) = L_j^0 \cup L_j^1$. Suppose that $\bigcup_{j \in J} |L_j^i|$ is isotopic to $|T|$ for both $i = 0, 1$; this can be achieved by taking a small enough neighborhood of each L_j . See Figure 8. We use $V(L_j)$ to denote a slightly larger tubular neighborhood satisfying this same condition. Fix an intersection point p of two elements $L_j, L_k \in T$. Denote by \mathcal{A}_p the connected component of the intersection of $U(L_j) \cap U(L_k)$ containing p . Similarly, let \mathcal{VA}_p be the connected component of the intersection of $V(L_j) \cap V(L_k)$ that contains p , and

denote $\mathcal{AA}_p = \mathcal{VA}_p \setminus \mathcal{A}_p$.

The open connected components of $U(T) \setminus \{\bigcup \mathcal{A}_p\}$ are homeomorphic to rectangles \mathcal{B}_i . Here the index p over the intersection points is assumed, as well as a suitable indexing for i . The third class of pieces constitute the interior of the complementary in \mathbb{CP}^1 of the open set formed by the union of the sets \mathcal{A}_p and \mathcal{B}_i ; its connected components are denoted \mathcal{C}_l . Thus, neighborhoods of the 0-cells, 1-cells and 2-cells are labeled \mathcal{A}_p , \mathcal{B}_i and \mathcal{C}_l respectively. See Figure 8.

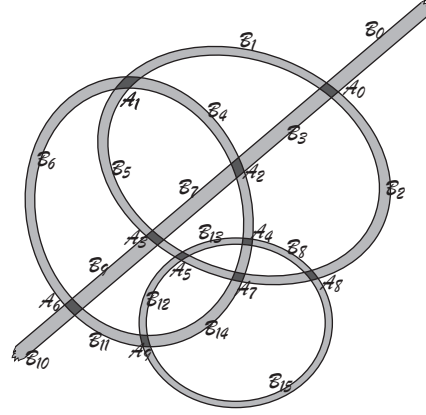


FIGURE 8. In dark gray the sets \mathcal{A}_p , in light gray the sets \mathcal{B}_i , associated to the subdivision of the figure 7.

Finally, we define the sets \mathcal{BB}_i . Let \mathcal{B}_i connect a couple of open sets⁵ of the form \mathcal{A}_p . There exists a curve $L_{\mathcal{B}_i}$ contained in \mathcal{B}_i which is a part of a curve $L_i \in T$; so $L_{\mathcal{B}_i}$ is a 1-cell in the decomposition associated to the adapted family T . Let $L_{\mathcal{B}_i}^0$ and $L_{\mathcal{B}_i}^\infty$ denote the two boundary components of \mathcal{B}_i which are part of the curves L_i^0 and L_i^1 defined above. Then we declare \mathcal{BB}_i^0 (resp. \mathcal{BB}_i^1) to be the connected component of $V(L_i) \setminus \mathcal{B}_j$ containing the boundary curve L_i^0 (resp. L_i^1). Their union $\mathcal{BB}_i^0 \cup \mathcal{BB}_i^1$ will be denoted \mathcal{BB}_i . See figures 9 and 10.

Once the required notation is established, we proceed to deform the structure to obtain vertical contact fibrations.

⁵Both sets may be the same for the self-intersecting curves.

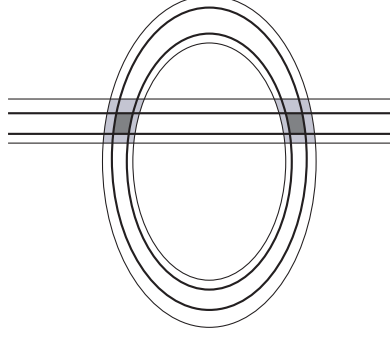


FIGURE 9. Example of two components \mathcal{VA}_p and \mathcal{VA}_q in light gray, containing \mathcal{A}_p and \mathcal{A}_q , in dark gray.

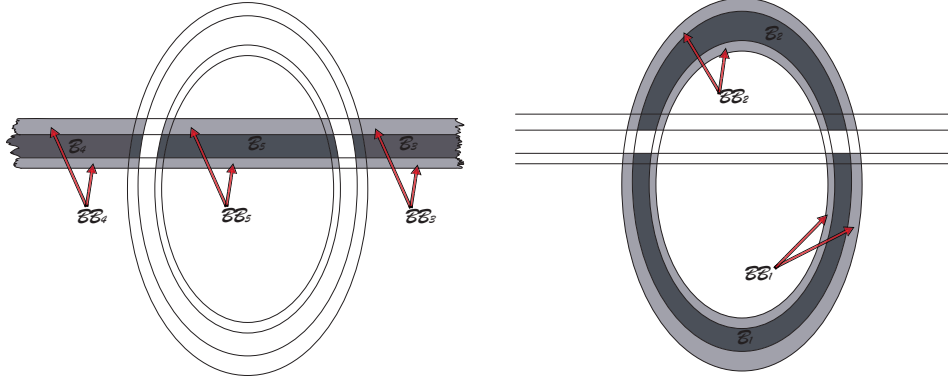


FIGURE 10. Example of the sets \mathcal{BB}_i for the subdivision of Figure 9.

6.4. The vertical construction. In the following it will be necessary to trivialize a given fibration *relative* to a given submanifold. This is provided in the next

Lemma 6.11. *Let $f : E \rightarrow B^2$ be a locally trivial smooth fibration over the disk with compact fibres. Assume that E has a smooth closed boundary ∂E . Suppose also that there is a collar neighborhood N of ∂E and a closed submanifold S such that restricting f to S, N or ∂E induces locally trivial fibrations. Let S_0, N_0 and E_0 be their fibres over $0 \in B^2$.*

Then there exists a diffeomorphism $g : E \longrightarrow E_0 \times B^2$ making the following diagram commute

$$\begin{array}{ccc} E & \xrightarrow{g} & E_0 \times B^2 \\ \pi \downarrow & & \downarrow \pi_0 \\ B^2 & \xlongequal{\quad} & B^2 \end{array}$$

such that $g(N) = N_0 \times B^2$ and $g(S) = S_0 \times B^2$.

Proof. Let g be Riemannian metric in E such that $(TE_z)^{\perp g} \subset TS$ as well as $(TE_z)^{\perp g} \subset T(\partial E)$, for the points z where the condition can be satisfied. Let $X = \partial_r$ be the radial vector field in $B^2 \setminus \{0\}$ and construct the connection associated to the Riemannian fibration:

$$H_\pi(e) = (T_e F_{\pi(e)})^{\perp g}.$$

The condition imposed on the Riemannian metric implies that ∂E and S are tangent to the horizontal connection H_π . Let \tilde{X} be a lift of X through H_π and $\phi_t(e)$ the flow of this vector field. Define

$$\begin{aligned} E & \xrightarrow{g} E_0 \times B^2 \\ e & \longmapsto (\phi_{(-\|\pi(e)\|)}(e), \pi(e)). \end{aligned}$$

This map satisfies the required properties. \square

We now prove the existence theorem of this section:

Proof of Theorem 6.2. We start by a blown-up fibration (f, C) . Recall that the exceptional divisor set is non-empty and there exists at least one exceptional divisor in the resulting almost contact fibration. Note that a *horizontal* connection H is defined away from $U(C)$ and provides the splitting specified in (5). Apply Proposition 6.7 to the family of distributions to ensure the existence of at least one overtwisted disk in each fibre. In particular we obtain \mathcal{B}_0 and \mathcal{B}_∞ .

Let T be an adapted family to the almost contact fibration such that $\partial \mathcal{B}_0$ and $\partial \mathcal{B}_\infty$ are both contained in two different 2-cells \mathcal{C}_0 and \mathcal{C}_∞ . Recall that in order to establish Theorem 6.2 we need to perform a deformation which is fixed in a neighborhood of $U(C)$ and which leaves the distribution H unchanged, i.e. it should be a strictly *vertical* deformation.

Deformation at the 0-cells: Let p be a 0-cells with neighborhood \mathcal{A}_p . The fibration trivializes over $\mathcal{V}\mathcal{A}_p$ and let (F_z, ξ_z) be the family of fibres and distributions; in case $\mathcal{V}\mathcal{A}_p$ is small enough the manifolds with boundary $\mathcal{F}_z = F_z \setminus (F_z \cap U(C))$ are all diffeomorphic⁶. Further, let N_z be a collar neighborhood of $\partial \mathcal{F}_z$ in which the distribution is contact. And let $U(E_j)_z$

⁶If $p \in \mathbb{CP}^1$ is the intersection point of two fake components of T then $U(C)$ does not intersect the fibres, hence this case is trivial.

be the intersection of $U(E_j)$ with the fibre \mathcal{F}_z . Applying the trivializing diffeomorphism provided in Lemma 6.11, we may assume $\mathcal{F}_z \times \mathcal{V}\mathcal{A}_p \cong \mathcal{F}$, $U(E_j)_z \times \mathcal{V}\mathcal{A}_p \cong U(E_j)$ and $N_z \times \mathcal{V}\mathcal{A}_p \cong N$.

We thus have: a manifold with boundary \mathcal{F} with a family of distributions ξ_z parametrized by the topological disk $K' = \mathcal{V}\mathcal{A}_p$ containing $K = \mathcal{A}_p$. Also a *good* set G of submanifold that are already contact for any parameter in K' , G consists of the union of N , $U(E_j)$ and a neighborhood of one of the two overtwisted disks⁷. Let us say a neighborhood of \mathcal{D}^∞ . A neighborhood of this set will not be perturbed. The remaining disk \mathcal{D}^0 is contactomorphic to the standard overtwisted disk for each element of the family of distributions. This set-up satisfies the hypothesis of Corollary 6.5 with $L' = \emptyset$. Since we are able to obtain a deformation relative to the boundary we may perform the deformation at each neighborhood of the 0-cells and extend trivially to the complement in \mathbb{CP}^1 .

Deformation at the 1-cells: Almost the same strategy applied to the 0-cells applies, although we should not undo the deformation in a neighborhood of the 0-cells. Fortunately Corollary 6.5 allows us to perform deformations relative to a subfamily, so in this case L' will be non-empty. See Figure 11. The reader is invited to precise the remaining details.

We have obtained the *vertical* contact condition in a neighborhood of the 1-cells and its intersections, and families parametrized by the boundary of these neighborhoods are fixed.

Deformation at the 2-cells: In this last situation the standard Eliashberg applies after a suitable trivialization of the smooth fibration provided by Lemma 6.11. The set L is a small tubular neighborhood of the boundary of the 2-cells. Except at \mathcal{C}_0 and \mathcal{C}_∞ , we may use any of the two families of overtwisted disks to apply the result. Let it be \mathcal{D}_z^0 . In the remaining family the distributions are contact and so we include it in the *good* set G . Moreover, the good set also contains N and $U(E_j)$. At \mathcal{C}_0 we use the family \mathcal{D}_z^0 , since it is the only one well defined over the whole set. Proceed analogously at \mathcal{C}_∞ . Note that this argument is possible because the deformation is relative to the boundary. Thus, after applying Theorem 6.3 to the 2-cells as indicated and extending trivially the deformation, we obtain a *vertical* contact distribution $(F_z, \tilde{\xi}_z)$ away from $U(C)$.

Finally, consider the direct sum $\tilde{\xi}_z \oplus H$ to include the critical set, which has not been deformed. This is the required *vertical* contact structure. Notice that this construction preserves the almost contact class of the distribution

⁷These disks are trivialized along with N using the previous lemma.

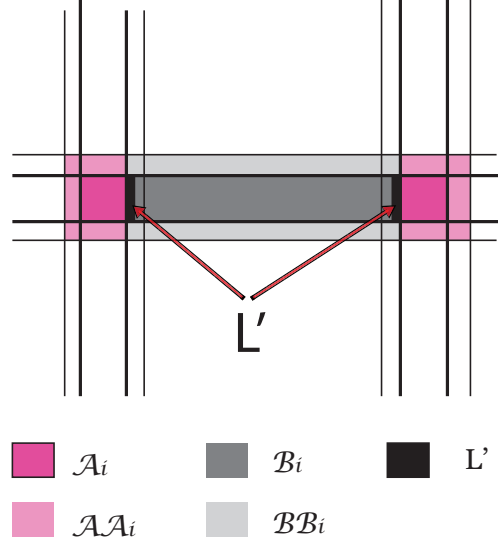


FIGURE 11. The set $L' \subset \mathcal{VB}_i$ is already a contact distribution.

since its performed homotopically in the *vertical* direction. \square

7. CONTACT STRUCTURE AWAY FROM THE \mathcal{C} -BALLS.

Given the initial almost contact distribution, we have reached through deformations a contact structure near the exceptional divisors and the critical set, and a *vertical* contact structure on the total space of the fibration. To obtain an honest contact structure the distributions have to be suitably changed in the *horizontal* direction. As in the previous section, this will be achieved in three stages. The content of this Section consists of the first two of these: deformation in the neighborhood of the 1-cells. The result of this procedure can be stated as follows:

Proposition 7.1. *Let (f, C) be the good almost contact fibration provided by Theorem 6.2. Then, there exists a horizontal deformation of the almost contact structure such that the resulting distribution is still a compatible almost contact structure and is a contact structure in the pre-image of a neighborhood of $f(C)$. Further, this deformation can be made relative to a neighborhood of the exceptional divisors and the critical set.*

The term *horizontal deformation* refers to the fact that the resulting distribution is still vertical; in this case, the vertical distribution is left fixed. The blown-up fibration (f, C) will not be perturbed to prove this fact, just the

almost contact structure.

The strategy will be to appropriately trivialize parts of the *vertical* contact fibration, the set of fibres over a neighborhood of the 0-cells to begin with. Afterwards we shall proceed with the 1-skeleton. In either situation we shall obtain a local model where the deformation is understood. Note that the local model will require a *vertical* contact structure on the fibres. The precise description is found ⁸ in the next:

Lemma 7.2. *Let $(F, \xi_{(s,t)})$ be a family of contact structures over the compact manifold F parametrized by $(s, t) \in [0, 1] \times [-\varepsilon, \varepsilon]$. Let $\alpha_{(s,t)}$ be the associated contact forms. On the total space of the fibration*

$$F \times [0, 1] \times [-\varepsilon, \varepsilon] \xrightarrow{\pi_t} F \times [0, 1],$$

we consider the distribution defined globally by the kernel of the form

$$\alpha_H(p, s, t) = \alpha_{(s,t)} + H(p, s, t)ds, \quad H \in C^\infty(F \times [0, 1] \times [-\varepsilon, \varepsilon]).$$

Suppose that $|H(p, s, t)| \leq c \cdot |t|$ and the distribution is constant along the t -lines, i.e. $\frac{\partial \alpha_{(s,t)}}{\partial t} = 0$. Assume that the 1-form α_H is already a contact form in a compact set G , and $\pi_t^{-1}(p, s) \cap G$ is connected.

Then, there is a small perturbation \tilde{H} of H relative to G such that $\alpha_{\tilde{H}}$ defines a contact structure. More formally, $|\tilde{H} - H| \leq 3c\varepsilon$ and $\tilde{H}|_G = H|_G$.

Proof. It will follow immediately from the contact condition on $\alpha = \alpha_H$:

$$d\alpha = d\alpha_{(s,t)} + dH \wedge ds \implies d\alpha^n = (d\alpha_{(s,t)})^n + (d\alpha_{(s,t)})^{n-1} \wedge dH \wedge ds.$$

Therefore, the contact condition is described as

$$d\alpha^n \wedge \alpha = (d\alpha_{(s,t)})^n \wedge Hds + (d\alpha_{(s,t)})^{n-1} \wedge \alpha_{(s,t)} \wedge dH \wedge ds > 0$$

Since $\frac{\partial \alpha_{(s,t)}}{\partial t} = 0$ the first term of the right hand side of the equation is zero,

$$d\alpha^n \wedge \alpha = (d\alpha_{(s,t)})^{n-1} \wedge \alpha_{(s,t)} \wedge \frac{\partial H}{\partial t} dt \wedge ds.$$

Thus, the condition to be a contact structure is

$$\frac{\partial H}{\partial t} > 0.$$

Thinking of $p \in F, s \in [0, 1]$ as *base* parameters, $\pi_t^{-1}(p, s)$ is a 4-parametric family of 1-dimensional manifolds. The connectedness of $\pi_t^{-1}(p, s) \cap G$ and the compactness of G assure that it is possible to perturb H to an \tilde{H} relative

⁸This lemma is an adapted version of results at Section 2.3 in [El], concerning deformations of a family of distributions near the 1 and 2-skeleta of the 3-manifold. The connectedness condition is stated there as the vanishing of a relative fundamental group.

to G and satisfying the contact condition; the connectedness condition allows us to perturb *at least* one end of a curve in $F \times [0, 1] \times [-\varepsilon, \varepsilon]$. \square

The *hamiltonian* perturbation just described will be performed after suitably trivializing the fibration over a disjoint set of 2-disks. The *vertical* contact condition, already achieved on the almost contact fibration, is naturally preserved in a local trivialization.

7.1. Deformation along intersection points. We need to achieve a contact structure in a neighborhood of the fibres over a neighborhood of the intersection points in T . First it is shown that a *vertical* relative trivialization can be performed near these points and that a further condition on the parallel transport can be assumed. Recall that the parallel transport along immersed curves induced by a contact connection is by contactomorphisms (see [Pr2]). That being said, the following lemmata will consist of exploiting the ideas involved in Lemma 6.11. We introduce the required notation.

Let z_k be an intersection of T , (ϕ_k, U_k) a chart centered at z_k with polar coordinates (r, θ) . Let $N_k = f^{-1}(U_k) \setminus U(f)$. We shall also denote by f the possible restrictions of the homonymous fibering map.

Denote by F_k the fibre over z_k , recall that $f : N_k \rightarrow U_k$ is a smooth fibration provided that the neighborhoods have been taken small enough, and thus the fibres are diffeomorphic to F_k . Restricting f to the boundary ∂N_k we also obtain a smooth fibration whose fibre is ∂F_k . The collar neighborhood theorem provides a neighborhood $U_{\partial N_k}$ of the boundary ∂N_k such that $f : U_{\partial N_k} \subset N_k \rightarrow U_k$ is a smooth fibration and the fibre U_{F_k} is diffeomorphic to $\partial F_k \times [0, \varepsilon]$. Let $\psi : U_{F_k} \rightarrow \partial F_k \times [0, \varepsilon]$ be such a diffeomorphism. Note that the almost contact fibration can be assumed to be a contact structure on $U_{\partial N_k}$ after Section 2.

The trivialization will be carried out using the flow of a lifted vector field. Because of this we work with a slight modification of the manifold F_k . Concretely, let $L \subset U_L$ be such that

$$\psi(L) = \partial F_k \times \{\varepsilon/2\}, \quad \psi(U_L) = \partial F_k \times [\varepsilon/2, 3\varepsilon/4]$$

Then $F_k \setminus \psi^{-1}(\partial F_k \times [0, \varepsilon/2])$ is a manifold with boundary L and collar neighborhood $U_L \subset U_{F_k}$. To ease notation we still call this *shortened* manifold F_k . In this situation, we are able to perform a trivialization respecting the *vertical* contact condition:

Lemma 7.3. *In the above mentioned setup with a sufficiently small neighborhood U_k of z_k , there exist a diffeomorphism G_k defined as*

$$G_k^{-1}(F_k \times B^2) \xrightarrow{G_k} F_k \times B^2, \quad G_k^{-1}(F_k \times B^2) \subset N_k$$

$$p \mapsto G_k(p) = (\varphi_{(-\|(\phi_k \circ f)(p)\|)}(p), (\phi_k \circ f)(p))$$

such that $(G_k)_*(\xi)$ is still a vertical contact structure with respect to the fibration over B^2 , and the family $\{\xi_z = (G_k)_*(\xi)|_{F_z}\}_{z \in B^2}$, is constant when restricted to the collar neighborhood $\psi(U_L) \times B^2$.

Proof. As in Lemma 6.11 we use a metric and its associated horizontal connection to trivialize the fibration $f|_{N_k} : N_k \rightarrow U_k$. Near $U_{\partial N_k}$ the distribution ξ is a contact structure, so we choose a metric such that the connection coincides there with the contact connection. Analogously, on the central fibre $f^{-1}(z_k)$ the metric connection should coincide with the almost contact connection associated to the original pair (ξ, ω) .

Let φ_t be the flow of the lifting of the radial vector field, $\varphi_t(p)$ is defined for $p \in N_k \setminus \partial N_k$ and a finite time depending on p . Note though that U_L is a compact submanifold so there is a uniform $\varepsilon_L > 0$ such that $\varphi_{-t}(q) \in L$ implies $q \in U_{N_k}$ for $q \in N_k$, $|t| \leq \varepsilon_L$. We define

$$G_k^{-1}(F_k \times B^2) \xrightarrow{G_k} F_k \times B^2, \quad G_k(p) = (\varphi_{(-\|(\phi_k \circ f)(p)\|)}(p), (\phi_k \circ f)(p))$$

where $\partial(G_k^{-1}(F_k \times B^2)) \subset U_{\partial N_k}$ and this boundary coincides with L at the central fibre. Note that the splitting of $(G_k)_*\xi$ must now be performed using $(G_k)_*\omega$. The connection has been chosen in order that the *vertical* factor of the distribution $(G_k)_*\xi$ is a contact structure once restricted to the fibres F_k ; since the family of distributions was already a contact distribution close to the boundary $L \times B^2$ the parallel transport is performed along contactomorphisms, so the family is indeed constant in the image of the collar neighborhood U_L . \square

In order to obtain the suitable local model of Lemma 7.2 we assure the existence of a deformation such that at least in one *direction* the parallel transport along the perturbed almost contact connection is a contactomorphism. Let I^2 denote a small closed rectangle⁹ with coordinates $(s, t) \in I^2$ and do note the slight abuse of notation we are about to commit using ξ for the *trivialized distribution*, once aware this should not lead to confusion.

Lemma 7.4. *Let $(F_k \times B^2, \xi)$ be the almost contact fibration obtained in Lemma 7.3. There exist a horizontal deformation of ξ , supported in the preimage of a small disk containing I^2 , such that the parallel transport along the lift of $\frac{\partial}{\partial s}$ through the perturbed almost contact connection is a contactomorphism.*

Remark 7.5. *As a consequence of Lemma 7.4 we can consider¹⁰ the contact structure near the central fibre to be given as the pull-back of the contact structure $\alpha_{(s,t)} + H(p, s, t)ds$ with $\frac{\partial \alpha_{(s,t)}}{\partial t} = 0$.*

⁹We could have worked with small disks from the beginning or just with rectangles but we consider more natural to prove the previous lemma with polar coordinates and have the local model in cartesian coordinates.

¹⁰Up to a fibre preserving diffeomorphism.

Proof. The almost contact distribution ξ splits as

$$\xi = \xi_v \oplus \xi_h$$

In order to provide a parallel transport through contactomorphisms, we focus on the central fibre and construct a distribution with the required property. For the first part, perform Moser's trick to each 1-parametric family $\xi_{(s,t)} = \ker \alpha_{(s,t)}$ where the t -coordinate is fixed. We obtain an s -family of diffeomorphisms

$$m_s^t : F_k \longrightarrow F_k, \quad (m_s^t)^* \xi_{(s,t)} = \xi_{(0,t)}$$

Note that these diffeomorphisms are supported away from U_L because the family is constant on U_L . Actually, Moser's argument can be made parametric and the family m_s^t can be chosen to behave smoothly with respect to the t -coordinate. We use this family of diffeomorphisms in the fibration, define

$$J_k : F_k \times I^2 \longrightarrow F_k \times I^2, \quad (p, s, t) \longmapsto (m_s^t(p), s, t)$$

and choose a vertical distribution $\tilde{\xi}_v$ at each fibre constant equal to the distribution on the central fibre. The condition $(m_s^t)^* \xi_{(s,t)} = \xi$ implies that $J_k^*(\xi_v) = \tilde{\xi}_v$.

We construct the appropriate distribution. Let $\tau_0 = \xi_h$ and τ_1 be the 2-distribution given by TB^2 inside $TF_k \oplus TB^2$; they are isotopic through horizontal distributions. Let τ_l be such an isotopy and $\chi : [0, \varepsilon) \longrightarrow [0, 1]$ be a smooth decreasing function such that

$$\chi(s) = 1, \text{ for } s \in [0, \varepsilon/4] \quad \chi(s) = 0, \text{ for } s \in [\varepsilon/2, \varepsilon].$$

The required distribution in a small rectangle of the trivialization is

$$\tilde{h}(p, s, t) = \tau_{\chi(s)}(p, s, t).$$

The parallel transport induced by the lift of the vector field $\frac{\partial}{\partial s}$ consists of the contactomorphisms on I^2 obtained through Moser's argument. Pushing-forward the construction to the manifold through G_k^{-1} we obtain the distribution as a direct sum of the vertical part and the push-forward of \tilde{h} . \square

Note that this lemma is a local result, not directly related to the almost contact fibration but to a trivialization of it. Gathering the previous results we can perform the *horizontal* deformation near the intersection points:

Corollary 7.6. *Let (f, C) be the almost contact fibration provided by Theorem 6.2. There exists a deformation of the blown-up vertical almost contact structure relative to $U(f)$ such that the resulting distribution is a contact structure on the preimages of a neighborhood of the intersection points in T .*

Proof. Both Lemma 7.3 and Lemma 7.4 provide a vertical almost contact structure $(\tilde{\xi}, \tilde{\omega})$ that coincides with the original almost contact distribution (ξ, ω) away from a neighborhood of the fibres over the intersection points

and in a neighborhood of the boundary ∂N_k . This almost contact structure is also homotopic to the original distribution on M , and parallel transport through *at least* one direction lifted by $f : N_k \rightarrow U_k$ to the almost contact connection consists of contactomorphisms.

In conclusion, there exists $\varepsilon > 0$, a neighborhood $V(z_k)$ of z_k and a map ψ_k such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}(V(z_k)) & \xrightarrow{\psi_k} & F_k \times B^2(\varepsilon) \\ \downarrow f & & \downarrow \\ V(z_k) & \xrightarrow{\phi_k} & B^2(\varepsilon) \end{array}$$

where the fibre of f is a contact manifold F_k with boundary. Since the map ψ_k is defined as a flow, maybe a smaller neighborhood is required but such neighborhood exists because of compactness. As remarked after Lemma 7.4, $(\psi_*)\xi$ satisfies the hypothesis of Lemma 7.2 for some small rectangle $I^2 \subset B^2(\varepsilon/2)$. Note that the set G can be taken empty for the contact condition is not *a priori* satisfied anywhere. Applying the lemma to the domain $F_k \times I^2$ and interpolating back to the almost contact structure $((\psi_k)_*\tilde{\xi}, (\psi_k)_*\tilde{\omega})$ in $F_k \times ((B^2(\varepsilon) \setminus B^2(\varepsilon/2)))$ we obtain an almost contact structure whose pull-back through ψ has the required properties after extending it to M by ξ . \square

Since we continue on perturbing, we still call this perturbed almost contact structure (ξ, ω) . Hence, at each stage this notation refers to the *most recent* perturbed almost contact structure.

7.2. Deformation along curves. Once we have achieved the contact condition in a neighborhood of the fibres over the 0-skeleton, we proceed with a neighborhood of the fibres over the 1-skeleton. Let \mathbb{T} denote a neighborhood of T .

We should be careful to maintain the contact structure over the neighborhood \mathbb{O} of the 0-cells. Let SS be a small neighborhood of the set of fibres over $\mathbb{T} \setminus \mathbb{O}$. See Figure 12 for the local picture of the domains of deformation. The argument applied over \mathbb{O} in the previous subsection works analogously when applied to SS ; thus, no detailed proofs will be given. The only subtlety lies in the appropriate choice of the compact set G for the application of Lemma 7.2.

Let $z, w \in \mathbb{CP}^1$ with corresponding neighborhood $\mathbb{O}_z, \mathbb{O}_w$; we focus on a line segment $S \subset |T|$ joining these two points. Let (ϕ, U) be a local chart¹¹ around $S \setminus (\mathbb{O}_z \cup \mathbb{O}_w)$ with cartesian coordinates (s, t) such that

$$\phi(U) = [0, 1] \times [-\varepsilon, \varepsilon], \quad \phi(S) = [0, 1] \times \{0\}.$$

¹¹The diffeomorphism ϕ will be orientation preserving.

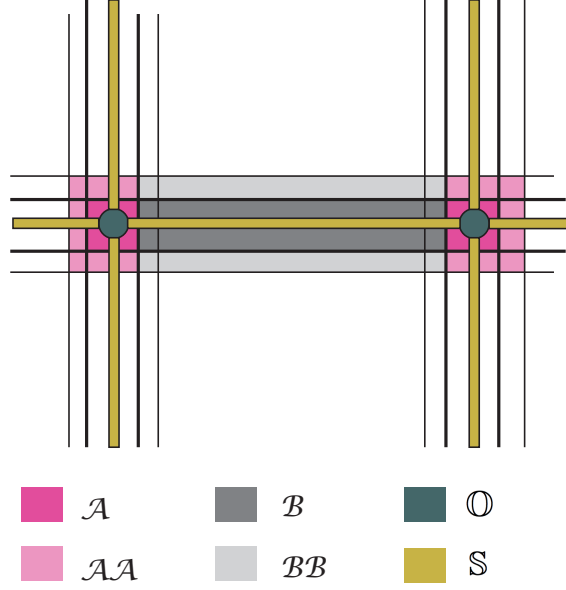


FIGURE 12. The deformation domains.

As we have established Lemma 7.4, we may assume the existence of a suitable trivialization. More formally, we shall use the following

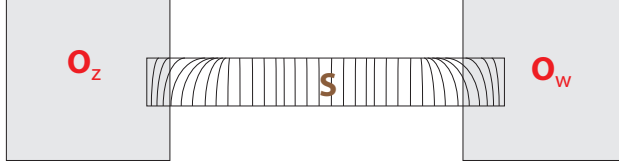
Lemma 7.7. *With the notation described above, there exist an arbitrarily small neighborhood \mathbb{S} of S and a horizontal deformation of the vertical almost contact structure (ξ, ω) supported in the pre-image of \mathbb{S} and relative the pre-image of $\mathbb{S} \cap \mathbb{O}_z$ and $\mathbb{S} \cap \mathbb{O}_w$ such that the distribution is still a contact structure on $U(f)$.*

Further, one may assume that the trivialization is such that the parallel transport of the associated almost contact connection along the vector field $\phi^* \left(\frac{\partial}{\partial t} \right)$ consists of contactomorphisms. \square

Note that the deformation is made relative to the pre-image of $\mathbb{S} \cap \mathbb{O}_z$ and $\mathbb{S} \cap \mathbb{O}_w$ for the almost contact distribution was already a contact structure.

The consequence is analogous to the previous subsection, mainly:

Corollary 7.8. *Let SS be a neighborhood of the fibres over the 1-skeleton associated to the family T . There exist a deformation of the vertical almost contact fibration (ξ, ω) , relative to small compact neighborhoods of the fibres over the 0-skeleton, such that the perturbed vertical almost contact distribution is a contact structure on SS .*

FIGURE 13. The deformation curves $\phi^* \left(\frac{\partial}{\partial t} \right)$.

Proof. Trivialize using Lemma 7.7. Be careful to choose the coordinates in the trivialization in such a way that the curves which provide the lift of $\phi^* \left(\frac{\partial}{\partial t} \right)$ has at most one of the ends in the fibres over the 0-skeleton (see Figure 13). This allows us to choose a compact set G containing the fibres over the two endpoints and satisfying the condition: the intersection of G with any such arc is connected. There might be the need to progressively shrink the neighborhoods of the fibres over the 0-skeleton. Apply lemma 7.2 to produce a contact structure in a neighborhood of the fibres over 1-skeleton without perturbing the existing contact structure in a small neighborhood of fibres over the endpoints. \square

These last two corollaries complete our argument,

Proof of Proposition 7.1. Just apply Corollaries 7.6 and 7.8. \square

8. BANDS THEORY.

The goal of this Section is to provide a model to fill the 2-cells. It will be constructed in Lemma 8.6. Once we are able to deform the almost contact structure in the interior of the 2-cells, we will complete the proof of the main results in the following Section.

The previous Sections have several arguments in which the dimension plays a crucial role. However most of them could be adapted to work in any dimension. It is our understanding that an appropriate generalization of Lemma 8.6 needs to be *found* in higher dimensions if one wants to prove the existence result in general. In this sense, this Section lies in the core of

the paper.

By a band fibration we mean a fibration $F \times [0, 1] \times \mathbb{S}^1 \longrightarrow [0, 1] \times \mathbb{S}^1$ over the annulus with the projection onto the second pair of factors. We declare an almost contact structure on $F \times [0, 1] \times \mathbb{S}^1$ to be adapted to the band fibration if it satisfies the following two conditions:

- It possesses an adapted form α admitting the following expression

$$\alpha = \alpha_0 + Hd\theta,$$

where α_0 is a contact form on F , H is a smooth function on the band $F \times [0, 1] \times \mathbb{S}^1$ and θ is the coordinate in \mathbb{S}^1 .

- The form α is a contact form close to the boundary $F \times \{0, 1\} \times \mathbb{S}^1$.

The purpose is to deform almost contact structures adapted to band fibrations into genuine contact structures. We begin by discussing a simple example, provided by Lemma 8.1, in which this can be easily done. It is not needed for the proof, but it clarifies the meaning of *deforming* the structure in the band.

Lemma 8.1. *Let $F \times [0, 1] \times \mathbb{S}^1 \longrightarrow [0, 1] \times \mathbb{S}^1$ be a band fibration. Let ξ be an adapted almost contact structure. Assume that $H(p, 1, \theta) - H(p, 0, \theta) > 0$ for all $p \in F$ and $\theta \in \mathbb{S}^1$. Then there is a deformation of ξ to a contact structure that preserves the structure in the boundaries.*

Proof. This is straightforward as the condition for α to be a contact form is just

$$\frac{dH}{dt} > 0.$$

Observe that the boundary condition allows us to deform the function H to attain the contact condition. \square

A contact fibration on $F \times [-A, A] \times \mathbb{S}^1$, for any $A > 0$, induced by the contact form

$$(6) \quad \alpha_A = \alpha_0 + td\theta$$

is an example of a band fibration. We call such a band fibration *standard*, it will be denoted M_A . The next result shows that any contact fibration admits an embedding into a suitable standard band.

Lemma 8.2. *Let $F \times [0, \varepsilon) \times \mathbb{S}^1$ be a manifold with a contact structure*

$$\xi = \ker(\alpha_0 + Hd\theta),$$

as above. Assume that H is a smooth function such that $\frac{dH}{dt} > 0$ and $|H| < A$, for some $A > 0$. Then, there is a strict¹² contact embedding of (M, ξ) in the standard contact band M_A .

¹²The application preserves the contact form.

Proof. The embedding is defined as

$$\begin{aligned} F \times [0, \varepsilon) \times \mathbb{S}^1 &\longrightarrow F \times [-A, A] \times \mathbb{S}^1 \\ (p, t, \theta) &\longrightarrow (p, H(p, t, \theta), \theta). \end{aligned}$$

□

We will relax a bit the notion of homotopy of almost contact structures in a way that will be enough for our purposes. The definition is natural and weaker than the usual one. There is a chance of proving the results with standard homotopies, however our argument performs a blow-down process at the end and it will only preserve an even weaker notion of homotopy, as we will see in Section 9.

Definition 8.3. Let (ξ_0, ω_0) and (ξ_1, ω_1) be two cooriented almost contact structures. They are *rough homotopic* if there is a continuous family of almost contact structures $\{(\widehat{\xi}_t, \widehat{\omega}_t)\}_{t \in [0, 1]}$ such that $(\xi_0, \omega_0) = (\widehat{\xi}_0, \widehat{\omega}_0)$, $\xi_1 = \widehat{\xi}_1$ and the bundle (ξ_1, ω_1) is symplectically isomorphic to the bundle $(\widehat{\xi}_1, \widehat{\omega}_1)$.

It is obvious that a homotopy of almost contact structures is a *rough* homotopy. What is not so clear is the converse. In fact, we will only get from this point *rough* homotopies and we will show at the end how to recover an honest homotopy out of them.

Now, we will prove in Lemma 8.6 that any distribution in a band fibration can be deformed, relative to the boundaries, to a new distribution that is contact. The way of proving it will be to replace the interior of the band by a *local model*. This model is provided in the following:

Proposition 8.4. Let (F, ξ) be a closed contact 3-manifold with $c_1(\xi) = 0$. Let $\delta > 0$ be any small constant. For any $A > 0$, there exist a contact band fibration $(B_A = F \times [0, 1] \times \mathbb{S}^1, \xi_A)$ with associated contact form β_A and two embeddings

$$e_1 : M_A \longrightarrow F \times [0, \delta] \times \mathbb{S}^1 \subset B_A \quad \text{and} \quad e_2 : M_A \longrightarrow F \times [1 - \delta, 1] \times \mathbb{S}^1 \subset B_A$$

satisfying the following properties:

- (i) $e_j^* \beta_A = \alpha_A$, for $j = 1, 2$ and α_A as in (6). i.e. the embeddings are strict contactomorphisms.

- (ii) The diffeomorphism

$$d_1 : [-A, A] \times \mathbb{S}^1 \longrightarrow [0, \delta] \times \mathbb{S}^1, \quad (t, \theta) \longmapsto \left((t + A) \frac{\delta}{2A}, \theta \right)$$

makes the following diagram commute

$$\begin{array}{ccc} M_A & \xrightarrow{e_1} & B_A \\ \downarrow & & \downarrow \\ [-A, A] \times \mathbb{S}^1 & \xrightarrow{d_1} & [0, \delta] \times \mathbb{S}^1 \end{array}$$

and equivalently for

$$\begin{aligned} d_2 : [-A, A] \times \mathbb{S}^1 &\longrightarrow [1 - \delta, 1] \times \mathbb{S}^1, \\ (t, \theta) &\longmapsto \left((1 - \delta) + (t + A) \frac{\delta}{2A}, \theta \right) \end{aligned}$$

there is a commutative diagram with the map e_2 .

- (iii) *Fixing a contact form α , associated to ξ , we have that the contact structure $(\xi_A, d\beta_A)$ is rough homotopic to the almost contact structure $(\xi \oplus T([0, 1] \times \mathbb{S}^1), d\alpha + dt \wedge d\theta)$ relative to the boundary.*

Proof. In order to find a band fibration we first construct a suitable fibration over the 2-sphere with fibre F and contact total space, then we will obtain the annulus base basically by removing two disks from this sphere and appropriately deform to a contact structure.

Use the fact that $c_1(\xi) = 0$ to find $\{X_1, X_2 \in \Gamma(\xi)\}$ a global framing of the contact distribution. Denote by X_0 the Reeb vector field associated to a contact form α_0 . Therefore $\{X_0, X_1, X_2\}$ is a global framing of TF . Let $\{\alpha_0, \alpha_1, \alpha_2\}$ be a dual framing. Denote the standard embedding of the 2-sphere as $e = (e_0, e_1, e_2) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$. It is a computation to verify that

$$\alpha_s = e_0 \cdot \alpha_0 + e_1 \cdot \alpha_1 + e_2 \cdot \alpha_2$$

is a contact form over $F \times \mathbb{S}^2$. The important properties for this to work are that $\{\alpha_0, \alpha_1, \alpha_2\}$ is a framing and the map e is a star-shaped embedding. Recall that $F_\infty = F \times (1, 0, 0)$ and $F_0 = F \times (-1, 0, 0)$ are contact sub-manifolds, moreover they are contactomorphic and the normal bundle of both of them is trivial. Take two copies of this contact manifold $X_1 = (F \times \mathbb{S}^2, \ker \alpha_s)$ and $X_2 = (F \times \mathbb{S}^2, \ker \alpha_s)$ and contact glue one to the other along the fibres F_∞^1 of X_1 and F_0^2 of X_2 . By Theorem 7.4.3 in [Ge], this provides a new fibration over \mathbb{S}^2 that we denote X' . Actually, the manifold X' comes equipped with a contact structure ξ' , such that the fibres F_0^1 and F_∞^2 remain contact and with trivial normal bundle.

We apply the normal neighborhood theorem to the contact submanifold F_0^1 to obtain a neighborhood U_0 of F_0^1 and a diffeomorphism

$$\phi_0 : U_0 \longrightarrow V_0 \subset F \times B^2(\varepsilon) \text{ such that } (\phi_0)_* \xi' = \ker(\eta_0),$$

where $\eta_0 = \alpha_0 + r^2 d\theta$ and (r, θ) are polar coordinates on $B^2(\varepsilon)$. So as to linearize the radius coordinate, fix the diffeomorphism

$$\begin{aligned} f : F \times ((\varepsilon/2)^2, \varepsilon^2) \times \mathbb{S}^1 &\longrightarrow F \times (\varepsilon/2, \varepsilon) \times \mathbb{S}^1 \\ (p, x, \theta) &\longrightarrow (p, \sqrt{x}, \theta). \end{aligned}$$

Indeed, this implies that

$$f^* \eta_0 = \alpha_0 + x d\theta.$$

To construct the annulus of radius $[0, \delta]$ and $[1 - \delta, 1]$ we add two domains X_0, X_∞ . To be precise, we perform a connected sum of the manifolds X'

with $X_0 = F \times (-\infty, \varepsilon^2) \times \mathbb{S}^1$ along $F \times ((\varepsilon/2)^2, \varepsilon^2) \times \mathbb{S}^1$ by using the contactomorphism f . Likewise, we do the same in a neighborhood of F_∞^2 to attach the strip $X_\infty = F \times (-\varepsilon^2, \infty) \times \mathbb{S}^1$ with contact form

$$(7) \quad \alpha_A = \alpha_0 + x d\theta.$$

This new contact manifold is diffeomorphic to $F \times (-\infty, \infty) \times \mathbb{S}^1$ and will be denoted by (Y, ξ_∞) with a fixed contact form β_∞ . Restricted to the domains X_0 and X_∞ the form β_∞ has the expression (7).

To achieve the fixed value A , it is left to increase the constant $\varepsilon > 0$ to conclude. For that, we define the k -covering

$$\begin{aligned} \varphi_k : Y &\longrightarrow Y \\ (p, t, \theta) &\longrightarrow (p, t, k\theta). \end{aligned}$$

Restricted to X_0 and X_∞ , the pull-back of the contact form is

$$\varphi_k^* \beta_\infty = \alpha_0 + k x d\theta.$$

Finally define $X'_0 = F \times (-\infty, k\varepsilon^2) \times \mathbb{S}^1$ and $X'_\infty = F \times (-k\varepsilon^2, \infty) \times \mathbb{S}^1$. We have an embedding map

$$\begin{aligned} \psi_0 : X'_0 &\longrightarrow X_0 \subset Y \\ (p, \theta, t) &\longrightarrow (p, \theta, t/k). \end{aligned}$$

Analogously there exists an embedding map $\psi_\infty : X'_\infty \longrightarrow X_\infty \subset Y$. The contact form β_∞ restricted to X'_0 and X'_∞ through ψ_0 and ψ_∞ after the k -covering becomes

$$(\varphi_k \circ \psi_j)^* \beta_\infty = \alpha_0 + x d\theta = \alpha_A.$$

We choose $k \in \mathbb{N}$ such that $k\sqrt{\varepsilon} > A$ and then M_A admits a strict embedding into X_0 and X_∞ . This embedding satisfies all the properties required in the statement.

Let us study the rough homotopic properties of this construction and deduce (iii). The constructed form can be written as

$$\beta_A = f_0 \alpha_0 + f_1 \alpha_1 + f_2 \alpha_2 + F dt + G d\theta,$$

where $f_1 = f_2 = 0$ and $f_0 = 1$ close to the boundaries. We can also assume that $(f_0, f_1, f_2) : [0, 1] \times \mathbb{S}^1 \longrightarrow \mathbb{R}^3$ is no-where vanishing. Observe that β_A can be deformed, through almost contact structures, to

$$\widehat{\beta}_A = f_0 \alpha_0 + f_1 \alpha_1 + f_2 \alpha_2,$$

with the same functions f_0, f_1 and f_2 . So, as long as the deformation concerns, we may and will consider $\widehat{\beta}_A$ instead of β_A . A word of caution is in order: since we are deforming the forms defining the distributions the homotopy studied next will not directly imply the relative property in the statement. Nevertheless, a deformation with a small hamiltonian can be homotoped back to the identity. Thus, the relative property will be recovered

after another homotopy in the boundary.

We have a global framing $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3 = dt, \alpha_4 = d\theta\}$. This allows us to trivialize the cotangent bundle of the band to produce a map

$$\begin{aligned} \varphi : T^*B_A \setminus \{0\} &\longrightarrow \mathbb{S}^4 \subset \mathbb{R}^5 \\ \eta = \sum_{j=0}^4 a_j \alpha_j &\longrightarrow \frac{(a_0, \dots, a_4)}{\|(a_0, \dots, a_4)\|} \end{aligned}$$

Consequently we can consider the map $\varphi(\widehat{\beta}_A) : B_A \longrightarrow \mathbb{S}^4 \subset \mathbb{R}^5$. Moreover, in this trivialization the *horizontal* almost contact structure

$$\xi \oplus T([0, 1] \times \mathbb{S}^1) = \ker \alpha_0$$

becomes the constant map $\varphi(\alpha_0) = (1, 0, 0, 0, 0)$. Recall that at every point of B_A , we have that $\ker \widehat{\beta}_A \not\subset TF$, i.e. the distribution never becomes vertical. So, we deduce that $\varphi(\widehat{\beta}_A)(p) \neq (0, 0, 0, 0, 1)$ for every $p \in B_A$. This implies that the map $\varphi(\widehat{\beta}_A)$ is not surjective and therefore is homotopic to the map $\varphi(\alpha_0)$, because the target manifold is a sphere. Therefore the associated distributions are homotopic as *cooriented* hyperplane fields. It is left to prove that the homotopy induces isomorphism of symplectic structures.

To understand the homotopy of the symplectic structures we need to be describe the forms more explicit. We proceed to geometrically perturb the form α_0 to $\widehat{\beta}_A$ in two steps: the intermediate form will be α_4 . First, define the homotopy of no-where vanishing 1-forms

$$\eta_t = (1 - t)\alpha_0 + t\alpha_4.$$

The initial dual framing $\{X_0, X_1, \dots, X_4\}$ restricts to the framing of $\ker \alpha_0 = \ker \eta_0$ provided by $\{X_1, X_2, X_3, X_4\}$. Recall that $\{X_1, X_2\}$ and $\{X_3, X_4\}$ are a symplectic basis associated to the *horizontal* almost contact structure. The strategy is pushing the symplectic basis through the homotopy to a new basis that we will declare symplectic again, defining in such a way a symplectomorphism between the first and the last bundle.

The framing of the distribution is pushed by the family η_t as

$$\{X_1, X_2, X_3, -tX_0 + (1 - t)X_4\},$$

as a family of framings for the family $\ker \eta_t$. Continue with the second part of the perturbation using the homotopy through the family

$$\eta'_t = (1 - t)\alpha_4 + t\widehat{\beta}_A.$$

In this family, the starting framing for $\ker \eta'_0$ is $\{X_1, X_2, X_3, -X_0\}$. We are going to extend the framing through a family of framings

$$\{X_1^t, X_2^t, X_3^t, X_4^t\}.$$

Let $X_3^t = X_3$ be constant and $X_4^t = tf_0X_4 - (1-t)X_0 + t(f_2X_1 - f_1X_2)$. It is left to compute the two first vectors of the family. For this we choose an auxiliary metric over TB_A satisfying that the basis $\{X_0, X_1, X_2, X_3, X_4\}$ is orthonormal. We choose the vectors X_1^t and X_2^t to be orthogonal for the metric with respect to X_3 and X_4^t . Moreover, they are selected in such a way that close to the boundary of B_A , where we know that $f_0 = 1$ and $f_1 = f_2 = 0$, they are constant. This is possible because they change only if X_3 and X_4^t change. So the induced symplectic structure is defined by declaring $\{X_1^1, X_2^1, X_3, f_0X_4 + f_2X_1 - f_1X_2\}$ to be a symplectic basis. Recall that the symplectic basis coincides with the original one close to the boundary of B_A . Therefore we have defined a symplectomorphism by

$$\begin{aligned} \Psi : \ker \alpha_0 &\longrightarrow \ker \widehat{\beta}_A \\ X_1 &\longmapsto X_1^1 \\ X_2 &\longmapsto X_2^1 \\ X_3 &\longmapsto X_3 \\ X_4 &\longmapsto X_4^1 \end{aligned}$$

inducing a symplectic structure $\widehat{\omega}$ on $\ker \widehat{\beta}_A$. We have to relate that symplectic structure with the one associated to $\ker \beta_A$ as a contact structure. Note that $\{X_3, X_4^1\}$ are a Lagrangian basis for $\{\ker \widehat{\beta}_A, d\widehat{\beta}_A\}$ and there is an arbitrarily small perturbation of this basis $\{Y_3, Y_4\}$ that becomes symplectic. Now we complete the symplectic basis to obtain a symplectic framing $\{Y_1, Y_2, Y_3, Y_4\}$ of $(\ker \widehat{\beta}_A, d\widehat{\beta}_A)$. Then

$$\begin{aligned} \Phi : (\ker \widehat{\beta}_A, \widehat{\omega}) &\longrightarrow (\ker \widehat{\beta}_A, d\widehat{\beta}_A) \\ X_1^1 &\longmapsto Y_1 \\ X_2^1 &\longmapsto Y_2 \\ X_3 &\longmapsto Y_3 \\ X_4^1 &\longmapsto Y_4 \end{aligned}$$

is the required symplectomorphism providing the rough homotopy. \square

Remark 8.5. *The symplectomorphism Φ is close to the identity near the boundary. This is because Y_3 is everywhere close to X_3 and X_4^1 coincides with X_4 near the boundary and X_4 is everywhere close to Y_4 . Since the framings are completed by symplectic orthogonality, they are close to each other near the boundary. So, up to small perturbations, the previous result can indeed be made relative to the boundary.*

It is not clear to us how to check that the symplectomorphism Φ is isotopic to the identity to produce a honest homotopy of almost contact structures.

The next lemma applies the previous proposition to deform a given almost contact distribution in a band fibration to honest contact structures. As previously stated, the result is essential in this *band theory*.

Lemma 8.6. *Let $F \times [0, 1] \times \mathbb{S}^1 \longrightarrow [0, 1] \times \mathbb{S}^1$ be a band fibration. Let $\xi_0 = \ker(\alpha_0 + Hd\theta)$ be an associated almost contact structure. Assume that F is 3-dimensional and $c_1(\ker(\alpha_0)) = 0$. Then there is a deformation ξ_1 of ξ_0 such that ξ_1 is a contact distribution rough homotopic to ξ_0 through a family of distributions preserving the structure in the boundaries.*

Observe that there is no longer a restriction of the values of the functions at the boundaries in the statement. Also note that the new contact structure is not compatible with the band anymore: there are fibres that will not be contact submanifolds. However, the fundamental property is that the deformation is supported away from the boundaries.

Proof. Apply Lemma 8.2 to embed the inner part of the band fibration

$$F \times [0, \varepsilon] \times \mathbb{S}^1 \longrightarrow [0, \varepsilon] \times \mathbb{S}^1$$

in a band M_A . Let φ_1 be such an embedding. Embed M_A in the domain B_A with the embedding e_1 in Proposition 8.4. Analogously we embed through e_2 the outer part of the band fibration

$$F \times [1 - \varepsilon, 1] \times \mathbb{S}^1 \longrightarrow [1 - \varepsilon, 1] \times \mathbb{S}^1$$

in B_A . By construction the hypersurface $(e_1 \circ \varphi_1)(F \times \{0\} \times \mathbb{S}^1)$ divides the manifold B_A in two disconnected domains. Denote the *inner* one by B_A^1 . The same construction applies to divide B_A by the hypersurface $e_2 \circ \varphi_2(F \times \{1\} \times \mathbb{S}^1)$. In that case let the *outer* side be B_A^2 . Then $B = B_A \setminus (B_A^1 \cup B_A^2)$ is a band fibration satisfying all the properties required in the statement.

It is left to check that the new contact structure is a rough deformation of the initial almost contact structure. But, this follows from property (iii) of the local model in Proposition 8.4. \square

The deformation provided in the previous lemma can be further described in the presence of a transverse knot in the initial fibre. This is particularly relevant because the set of exceptional divisors intersect the fibres in transverse links. The result that we will use is a consequence of the lemma:

Corollary 8.7. *In the same setup than Lemma 8.6, let $K \simeq \mathbb{S}^1 \longrightarrow (F, \xi_0)$ be a transverse knot. Let $U(K)$ be a neighborhood of K in F admitting a trivialization chart*

$$\Psi : U(K) \longrightarrow K \times B^2(\varepsilon)$$

whose coordinates are (z, r, ν) . Suppose that ξ restricted to $U_B = U(K) \times [0, 1] \times \mathbb{S}^1$ can be written, trivializing with Ψ , as

$$\xi = \ker(dz + r^2 d\nu + Hd\theta)$$

for some smooth function H with $|H(z, 0, 0, 0, \theta)|$ and $|H(z, 0, 0, 1, \theta)|$ sufficiently small functions. Then the deformation ξ_1 of ξ_0 provided in the

previous lemma is a distribution such that $K \times [0, 1] \times \mathbb{S}^1$ with ξ_1 restricted to it is a contact submanifold. Further, the contact structure is precisely

$$(8) \quad \xi_1 = \ker(\cos(2\pi t)dz + t \sin(2\pi t)d\theta).$$

Proof. We begin proving the case in which

$$H(z, 0, 0, 0, \theta) = H(z, 0, 0, 1, \theta) = 0.$$

With this assumption the embeddings e_1 and e_2 in Lemma 8.6 are trivial close to the knot. Therefore we are in the local model of Proposition 8.4. More precisely, we need to understand the structure in $X' \setminus (F_0 \cup F_\infty)$. Observe that in the chosen trivialization Ψ the framing restricted to $U(K)$ becomes $\{\alpha_0 = dz + r^2 d\nu, \alpha_1 = dx, \alpha_2 = dy\}$, where the coordinates (x, y) are the cartesian coordinates of (r, ν) :

$$\begin{aligned} x &= r \cos \nu, \\ y &= r \sin \nu. \end{aligned}$$

Without loss of generality the initial embedding of the sphere

$$e = (e_1, e_2, e_3) : \mathbb{S}^2 \longrightarrow \mathbb{R}^3$$

in Proposition 8.4 can be chosen to be the spherical coordinates embedding:

$$(9) \quad \begin{aligned} e_0(t, \theta) &= \cos(\pi t), \\ e_1(t, \theta) &= \sin(\pi t) \cos(2\pi \theta), \\ e_2(t, \theta) &= \sin(\pi t) \sin(2\pi \theta), \end{aligned}$$

with $t \in [0, 1]$, $\theta \in [0, 1]$. The contact connected sum is equivalent to changing the embedding by the following one

$$\begin{aligned} e_0^2(t, \theta) &= \cos(2\pi t), \\ e_1^2(t, \theta) &= \sin(2\pi t) \cos(2\pi \theta), \\ e_2^2(t, \theta) &= \sin(2\pi t) \sin(2\pi \theta), \end{aligned}$$

and keep the contact structure to be

$$\alpha_s^2 = e_0^2 \alpha_0 + e_1^2 \alpha_1 + e_2^2 \alpha_2.$$

The k -covering is tantamount to changing the map e to

$$\begin{aligned} e_0^{2,k}(t, \theta) &= \cos(2\pi t), \\ e_1^{2,k}(t, \theta) &= \sin(2\pi t) \cos(2k\pi \theta), \\ e_2^{2,k}(t, \theta) &= \sin(2\pi t) \sin(2k\pi \theta), \end{aligned}$$

and the final form over B_A –without the completion of the strips X_0 and X_∞ – is defined as

$$\alpha_s^{2,k} = e_0^{2,k} \alpha_0 + e_1^{2,k} \alpha_1 + e_2^{2,k} \alpha_2.$$

Substituting we obtain

$$\alpha_s^{2,k} = \cos(2\pi t)\alpha_0 + \sin(2\pi t)\cos(2k\pi\theta)\alpha_1 + \sin(2\pi t)\sin(2k\pi\theta)\alpha_2.$$

We slightly perturb the form to

$$\begin{aligned} \alpha_s^{2,k}(\delta) &= \cos(2\pi t)\alpha_0 + \sin(2\pi t)\cos(2k\pi\theta)\alpha_1 \\ &\quad + \sin(2\pi t)\sin(2k\pi\theta)\alpha_2 + \delta b(r)t\sin(2\pi t)d\theta, \end{aligned}$$

with a function $b : [0, \varepsilon] \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying the conditions

- $b(r) = 1$, for $r \leq \varepsilon/3$,
- $b(r)$ is decreasing,
- $b(r) = 0$, for $r > 2\varepsilon/3$.

For any sufficiently small δ – by Gray’s stability – the perturbation does not change the global contact structure, since it can be made arbitrarily C^1 –small. Now we restrict to $K \times [0, 1] \times \mathbb{S}^1$ to obtain the expected formula (8), up to a change of coordinates.

To conclude the general case, that is $|H(z, 0, 0, \theta, 0)|$ and $|H(z, 0, 0, \theta, 1)|$ sufficiently small, we just need to slightly modify the argument. We replace the formulas (9) by

$$\begin{aligned} e_0(t, \theta) &= h_1(\pi t), \\ e_1(t, \theta) &= h_2(\pi t)\cos(2\pi\theta), \\ e_2(t, \theta) &= h_2(\pi t)\sin(2\pi\theta), \end{aligned}$$

where $h_1, h_2 : [0, 1] \rightarrow \mathbb{R}^2$ are smooth functions satisfying that

$$h_1 h_2' - h_2 h_1' > 0,$$

and the map $(h_1, h_2) : [0, 1] \rightarrow \mathbb{R}^2$ does not contain the point $(0, 0)$ as part of its image; note that it winds around the origin once as shown in Figure 14.

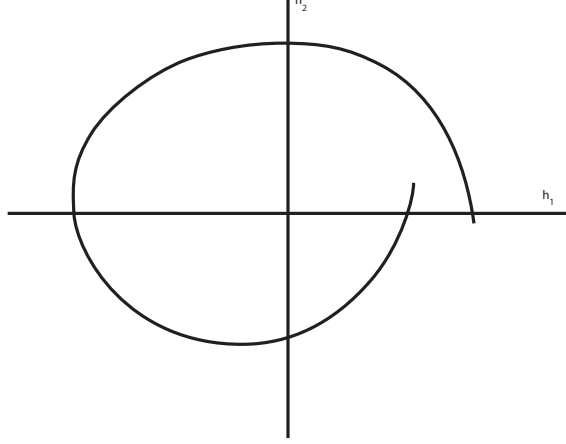
□

9. THE END: FILLING THE HOLES AND BLOWING-DOWN.

We gather all the previous arguments to conclude the proof of the Theorems 1.1 and 1.2. The steps consist of filling the interior of the 2–cells, deforming the structure in exceptional divisors to be able to obtain a contact structure in the initial manifold and proving the invariance of the first Chern class under the process.

9.1. Filling the holes.

Theorem 9.1. *Let (f, C) be the blown-up fibration and let ξ be the adapted almost contact distribution provided by Proposition 7.1. There exists a contact distribution ξ' rough homotopic ξ . Further, the restriction of ξ' to each exceptional sphere induces the unique overtwisted contact structure homotopic to the standard one.*

FIGURE 14. The map (h_1, h_2) .

Proof. Recall that the almost contact structure is vertical contact, and contact away from a disjoint union of preimages of balls $B_i \subset \mathbb{CP}^1$. We assume that ξ is a contact structure in the fibres close to the boundary of B_i , by slightly enlarging the balls if necessary. Moreover the restriction of f to the preimages of each B_i is a smooth fibration, since the critical points of f are in the complement of the balls. We will produce a deformation over each B_i supported away from the boundary, so it will extend to a global deformation.

Choose a diffeomorphism $\phi_i : B_i \rightarrow B^2(1 + \varepsilon)$ (ε is a fixed small positive constant). Observe that the diameter of the balls B_i can be chosen as small as needed by adding extra fake components. This allows us to assume that the inverse of ϕ_i has arbitrarily small derivatives. We define the map

$$g = \phi_i \circ f : U_i \rightarrow B^2(1 + \varepsilon), \quad \text{where } U_i = f^{-1}(B_i).$$

We may assume that $g^{-1}(B^2(1 + \varepsilon) \setminus B^2(1))$ is an open set where ξ is already contact. We have to deform the interior, since the boundary has already been deformed in the previous sections.

Trivialize the fibration g using Lemma 7.3. So we get a map

$$G : U_i \rightarrow F_0 \times B^2(1 + \varepsilon), \quad \text{such that } G_*\xi = \ker(\alpha_0 + Hd\theta).$$

Close to the exceptional divisors $\{E_j\}$ the distribution has not been deformed, therefore the restriction of the map G to an arbitrarily small neighborhood of $\{E_j\}$ does not depend on the chosen adapted family. Hence,

close to $\{E_j\}$ the norm of H is only proportional to the C^1 -norm of ϕ_i^{-1} . Consequently we may assume that $|H| \leq \delta$ for any $\delta > 0$ in the neighborhood of the exceptional divisors.

Now we perturb the hamiltonian H defined on $F_0 \times B^2(1 + \varepsilon)$ to a new \tilde{H} . This perturbation is localized in the area $F_0 \times B^2(2\varepsilon)$ and it is defined by imposing:

- (i) $\tilde{H}(p, r, \theta) = r^2$, for $r \leq 3\varepsilon/2$,
- (ii) $|\tilde{H}| \leq \varepsilon$, for $r \leq 2\varepsilon$,
- (iii) $\tilde{H} - H = 0$, for $r \geq 2\varepsilon$.

Hence we have an almost contact structure $\tilde{\xi} = \ker(\alpha_0 + \tilde{H}d\theta)$ adapted to the band $F_0 \times [\varepsilon, 1 + \varepsilon] \times \mathbb{S}^1$. We can apply the Proposition 8.4 to this band to find a new contact structure $\tilde{\xi}'_0$ over $F_0 \times [\varepsilon, 1 + \varepsilon] \times \mathbb{S}^1$ that matches $\tilde{\xi}$ close to the boundary of the band and that is rough homotopic to it relative to the boundary. Thus, we can pull it back to $\xi' = G^*(\tilde{\xi}'_0)$ in $g^{-1}((\varepsilon, 1 + \varepsilon) \times \mathbb{S}^1)$ and it easily extends to a contact distribution over $g^{-1}(B(1 + \varepsilon))$ by replacing ξ by $\tilde{\xi}$ over $g^{-1}(B(\varepsilon))$. Observe that this deformation naturally extends to the whole manifold.

Finally we are also in the hypothesis of Corollary 8.7, since the image through G of the exceptional divisors is of the form $L \times B^2(1 + \varepsilon)$, where L is a transverse link. Moreover, by the previous discussion, the function \tilde{H} can be chosen to have arbitrarily small norm in the neighborhood of the divisors. This implies that every exceptional sphere is deformed by a full Lutz twist over each U_i . Recall that an arbitrary number of full Lutz twists does not change the homotopy type of the distribution and it is therefore, by Eliashberg classification result, isotopic to the standard overtwisted structure. \square

9.2. Interpolation at the exceptional divisors. We have obtained the contact condition in the blown-up manifold, it is left to blow-down. Recall that the exceptional spheres have the standard tight contact structure when they appeared as exceptional divisors, hence the contact blow-down procedure cannot be performed directly. The purpose is thus to deform the contact distribution on a neighborhood of the exceptional spheres. We shall modify the contact structure on them to be the standard one, then they will be ready to blow-down.

The precise statement of the above discussion is the content of the following:

Theorem 9.2. *Let $(\mathbb{S}^3 \times B^2(\varepsilon), \xi_0)$ have contact form*

$$(10) \quad \eta = \alpha_{ot} + r^2 d\theta,$$

where α_{ot} is a contact form associated to the unique overtwisted contact structure homotopic to the standard contact structure on \mathbb{S}^3 .

Then, there exists a deformation ξ_1 of ξ_0 supported in $\mathbb{S}^3 \times B^2(\varepsilon/2)$ such that the ξ_1 is a contact structure and $\mathbb{S}^3 \times \{0\}$ inherits the standard contact structure.

This result is a straightforward consequence of the following result

Lemma 9.3 (Lemma 3.2 in [EP]). *There exists a contact structure on $\mathbb{S}^1 \times B^2(\varepsilon) \times B^2(\delta)$ satisfying the following properties:*

- (i) *on $\mathbb{S}^1 \times B^2(\varepsilon/2) \times B^2(\delta)$ the contact structure is $\ker\{\alpha_{std} + r^2 d\theta\}$, on $\mathbb{S}^1 \times (B^2(\varepsilon) \setminus B^2(3\varepsilon/4)) \times B^2(\delta)$ is given by $\ker\{\alpha_{ot} + r^2 d\theta\}$, and on $\mathbb{S}^1 \times \partial(B^2(\varepsilon) \times B^2(\delta))$ is given by $\ker\{\alpha_{std} + r^2 d\theta\}$.*
- (ii) *The contact structure is homotopic relative to the boundary – as an almost contact distribution – to the standard contact structure on $\mathbb{S}^1 \times B^2(\varepsilon) \times B^2(\delta)$.*

Proof of Theorem 9.2. We may assume that the standard overtwisted contact structure is standard on the neighborhood of the complement of the unknot. Hence we can glue the model provided by Lemma 9.3 to produce the required contact structure. \square

Notice that the deformation will be through almost contact structures and so it is an actual homotopy of almost contact structures, not just a *rough* one.

9.3. Ending and controlling the homotopy. Finally, the proof of Theorems 1.1 and 1.2 will be completed.

The starting point is the blown-up fibration and the contact structure constructed in Theorem 9.1. The construction provides the standard overtwisted structure on the exceptional spheres since a sequence of full Lutz twists are performed. By the standard neighborhood Theorem, there is a neighborhood U_i of each exceptional sphere E_i and a diffeomorphism

$$\phi_i : U_i \longrightarrow \mathbb{S}^3 \times B^2(\varepsilon) \text{ such that } \xi = \ker \phi_i^* \eta,$$

for some small $\varepsilon > 0$ and with η the contact form appearing in formula (10). Denote $\xi_0 = \ker \eta$ and apply Theorem 9.2 to obtain a new contact structure ξ_1 over $\mathbb{S}^3 \times B^2(\varepsilon)$. Therefore we can define $\xi' = \phi_i^* \xi_1$ over U_i . Notice that since it coincides with ξ close to the boundary of U_i we can globally define ξ' by extending it through ξ wherever is not defined.

The distribution ξ' is rough homotopic to our initial almost contact blown-up structure. Recall that Lemma 5.6 allows us to blow-down along the exceptional divisors to obtain a new contact structure over the initial manifold. It is only needed to keep track of the choice of framing determined by the integer k in the blow-up construction and use the same k in the blow-down process. Thus, we have produced a contact structure in our initial manifold.

This concludes the proof of the existence part in the statements.

Let us briefly review the argument used so far. We started with an almost contact structure ξ on the 5-dimensional manifold M . A slight perturbation of the almost contact structure is performed and then we proceed with contact blowing-up in order to obtain an almost contact structure $\tilde{\xi}$ over a new manifold \tilde{M} . Afterwards, we produce a rough homotopic distribution $\tilde{\xi}_1$ from $\tilde{\xi}$ that we are able to blow-down, since we ensured that it coincides with the initial $\tilde{\xi}$ over the exceptional divisors. This produces a contact structure ξ_1 over M .

There exists a topological relation between ξ and ξ_1 as stated in the classification part of Theorems 1.1 and 1.2. This relation is the content of the following

Lemma 9.4. *In the above notation, $c_1(\xi) = c_1(\xi_1)$.*

Proof. The starting hypothesis is that $\tilde{\xi}$ and $\tilde{\xi}_1$ are *rough* homotopic. This in particular implies that there is a symplectic bundle isomorphism

$$\begin{array}{ccc} \tilde{\xi} & \xrightarrow{\tilde{\Psi}} & \tilde{\xi}_1 \\ \downarrow & & \downarrow \\ \tilde{M} & \xrightarrow{id} & \tilde{M} \end{array}$$

Restricting $\tilde{\Psi}$ to $M \setminus (\bigcup E_i)$ we get a symplectic isomorphism defined as

$$\begin{array}{ccc} \xi & \xrightarrow{\Psi} & \xi_1 \\ \downarrow & & \downarrow \\ M \setminus B & \xrightarrow{id} & M \setminus B \end{array}$$

By fixing complex structures compatible with the symplectic structures of the bundles we may assume that Ψ is a complex isomorphism of complex bundles. We need to extend it to a global isomorphism, over the whole M . This would produce a *rough* homotopy of the almost contact structures implying the statement, however this is not true in general as we will explain later on.

We will prove a weaker statement by inducing a complex isomorphism of the complex determinant bundles denoted as

$$det(\Psi) : \Lambda^2 \xi \longrightarrow \Lambda^2 \xi_1,$$

this also implies the statement for the determinant bundles prescribe the first Chern class. The aim is now to extend an isomorphism of complex line bundles that is defined away from a disjoint union of circles. Fortunately,

this can be done in general. To prove so, we restrict ourselves to the neighborhood of a circle $\mathbb{S}^1 \subset B$, say $U \cong \mathbb{S}^1 \times B^4$. The line bundles are trivial over such a neighborhood for homotopy reasons, therefore the isomorphism $\det(\Psi)$ is a map

$$\mathbb{S}^1 \times (B^4 \setminus \{0\}) \longrightarrow \mathbb{S}^1.$$

Thus to extend $\det(\Psi)$ is a matter of extending the map to $\mathbb{S}^1 \times B^4$. By standard obstruction theory ([Ha], Corollary 4.73), we have that the obstruction is described by a sequence of cohomology elements lying in

$$H^{i+1}(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \pi_i(\mathbb{S}^1)).$$

These are trivial except for $i = 1$. Therefore we need to compute

$$H^2(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \pi_1(\mathbb{S}^1)) = H^2(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \mathbb{Z}),$$

applying Lefschetz duality we conclude that

$$H^2(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \mathbb{Z}) = H_3(\mathbb{S}^1 \times B^4, \mathbb{Z}) = 0$$

and thus the remaining obstruction class necessarily vanishes. \square

This completes the proof of Theorem 1.2. We conclude the proof of Theorem 1.1 using the following auxiliary:

Lemma 9.5 ([Ham], Theorem 8.18). *The obstruction for a pair of almost contact structures ξ and ξ_1 to be homotopic as almost contact structures in a 5-dimensional manifold is provided by an element $d \in H^2(M, \mathbb{Z})$. Moreover, we have that $2d = c_1(\xi_1) - c_1(\xi)$.*

Assume that there are no 2-torsion elements in $H^2(M, \mathbb{Z})$, the above lemma applied to our two distributions implies that the obstruction vanishes. Hence, they are homotopic. This concludes the proof of Theorem 1.1.

Remark 9.6. *The obstructions to the extension of the bundle map Ψ in the previous proof to obtain a full isomorphism are elements of*

$$H^{i+1}(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \pi_i(U(2))).$$

In particular for $i = 3$ the obstruction class lies in

$$H^4(\mathbb{S}^1 \times B^4, \mathbb{S}^1 \times (B^4 \setminus \{0\}), \mathbb{Z}) = H_1(\mathbb{S}^1 \times B^4, \mathbb{Z}) = \mathbb{Z}$$

and we cannot directly conclude the vanishing of the obstruction class.

The uniqueness of a contact structure in every homotopy class of almost contact structures does not hold in a 5-fold. There are examples in the literature, for instance [NK] proves that every fillable contact structure has a non-fillable contact structure in the same almost contact homotopy class.

The construction described in this article requires a fair amount of choices. Though, the dependence of the contact structure with respect to them may be understood. The three main ingredients are the stabilization procedure of almost contact pencils, in the same spirit than Giroux's stabilization for a contact open book decomposition [Gi]; the addition of fake curves in the

triangulation increasing the amount of holes filled with the local model and the canonicity of the contact blow-up procedure.

10. NON-COORIENTABLE CASE

10.1. Definitions. Let M^{2n+1} be a closed manifold, not necessarily orientable. In order to state the Theorem 1.1 in the non-coorientable setting, we need to give a definition of non-coorientable almost contact structure. This will be a distribution with a suitable reduction of the structure group along with a property requiring a relation between the normal bundle and the distribution. First we introduce the Lie group $\mathfrak{A}(n)$ defined as

$$\mathfrak{A}(n) = \{A \in O(2n) : AJ = \pm JA\}, \quad \text{where } J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$$

Notice the following properties:

1. The group $\mathfrak{A}(n)$ has two connected components. Actually, it is homeomorphic to $U(n) \times \mathbb{Z}_2$.
2. Its group structure is isomorphic to a semidirect product $U(n) \rtimes_{\rho} \mathbb{Z}_2$.

More precisely, let $\mathbb{I} = \begin{pmatrix} Id_n & 0 \\ 0 & -Id_n \end{pmatrix}$, then the action

$$\rho : \mathbb{Z}_2 \longrightarrow \text{Aut}(U(n)), \quad a \longmapsto (U \longmapsto \mathbb{I}^a U \mathbb{I}^a)$$

induces the semidirect product structure in the usual way.

3. There is a natural group morphism $\mathfrak{s} : \mathfrak{A}(n) \longrightarrow \mathbb{Z}_2$ defined as

$$\mathfrak{s}(A) = \text{tr}(JAJ^{-1}A^{-1})/(2n),$$

i.e. under the previous isomorphism, \mathfrak{s} is the projection onto the second factor of $U(n) \rtimes_{\rho} \mathbb{Z}_2$.

We now deduce topological implications of the existence of a contact structure. Let $\xi \subset TM$ be a –possibly non-coorientable– contact structure on M with a fixed set $\{U_i\}$ of trivializing contractible charts. Choose α_i as a local equation for $\xi|_{U_i}$, then

$$\alpha_i = a_{ij}\alpha_j, \quad \text{with } a_{ij} : U_i \cap U_j \longrightarrow \{\pm 1\}.$$

This implies that $\{a_{ij}\}$ are the transition function of the normal line bundle TM/ξ . Further, $(d\alpha_i)|_{\xi} = a_{ij}(d\alpha_j)|_{\xi}$. In particular, we may choose a family of compatible complex structures $\{J_i\}$ for the bundle ξ satisfying $J_i = a_{ij}J_j$.

First, note that there is a group injection

$$\mathfrak{A}(n) \longrightarrow O(2n+1), \quad A \longmapsto \begin{pmatrix} A & 0 \\ 0 & \mathfrak{s}(A) \end{pmatrix}$$

and thus the structure group of M reduces to $\mathfrak{A}(n)$. And second, a $\mathfrak{A}(n)$ –bundle E induces via the morphism \mathfrak{s} a real line bundle $\mathfrak{s}(E)$. This construction applied to ξ gives the line bundle TM/ξ in the case above. These two properties will be the ones required in the following:

Definition 10.1. *An almost contact structure on a manifold M^{2n+1} is a codimension 1 distribution $\xi \subset TM$ such that the structure group of ξ reduces to $\mathfrak{A}(n)$ and $\mathfrak{s}(\xi) \cong TM/\xi$.*

Observe that the definition for a cooriented almost contact distribution coincides with the one previously given. There are some immediate topological consequences of the existence of such a ξ . Indeed:

- (i) If n is an even integer, then $\mathfrak{A}(n) \subset SO(2n)$. In particular, the distribution ξ is oriented.
- (ii) If n is an even integer, there is an isomorphism

$$(11) \quad TM/\xi \cong \det(TM).$$

Hence, any almost contact structure in an orientable 5-dimensional manifold is cooriented.

- (iii) If n is an odd integer, then $\mathfrak{s} = \det$ as morphisms from $\mathfrak{A}(n)$ to \mathbb{Z}_2 . Therefore M is orientable since

$$\det(TM) \cong \det(\xi \oplus (TM/\xi)) \cong \det(\xi) \otimes \mathfrak{s}(\xi) \cong \det(\xi)^2 \cong \mathbb{R}$$

Let M^{2n+1} be a non-orientable manifold with n an even integer. Then there exists a canonical $2 : 1$ cover

$$\pi_2 : M_2 \longrightarrow M$$

satisfying the following properties:

1. M_2 is an orientable manifold.
2. Any almost contact structure ξ on M lifts to an almost contact structure $\pi_2^*\xi$ on M_2 . Moreover, such a distribution is cooriented because of equation (11).

The existence of π_2 allows us to define the i -th Chern class of an almost contact distribution ξ on M as $c_i(\xi) := c_i(\pi_2^*\xi) \in H^{2i}(M_2, \mathbb{Z})$. These classes are only defined if n is even.

10.2. Statement of the main result. We are in position to state the equivalent result of Theorem 1.2 in the non-coorientable setting:

Theorem 10.2. *Let M be a non-orientable closed 5-dimensional manifold. Given an almost contact structure ξ with first Chern class $c_1(\xi)$, there exists a contact structure ξ' such that $c_1(\xi') = c_1(\xi)$.*

Any further statement on the homotopy class of the distribution would require an appropriate version of Lemma 9.5.

Proof. Let $\pi_2 : (M_2, \pi_2^*\xi) \longrightarrow (M, \xi)$ be the double cover mentioned above. We claim that most of the constructions developed in this article can be performed \mathbb{Z}_2 -invariantly. Let us briefly discuss it:

- (i) An almost contact pencil (f, B, C) can be made \mathbb{Z}_2 -invariant, to be precise B and C are \mathbb{Z}_2 -invariant subsets and f is \mathbb{Z}_2 -invariant as a map. In particular the action preserves the fibres. This is because the approximately holomorphic techniques can be developed in that setting. See [IMP] for the details of the construction in the \mathbb{Z}_2 -invariant setting.
- (ii) The deformations performed in Section 4 can easily be done in a \mathbb{Z}_2 -invariant way. Also, the contact blow-up along a \mathbb{Z}_2 -invariant loop can be built to preserve that symmetry.
- (iii) Subsection 6.2 is also prepared for the \mathbb{Z}_2 -invariant setting. Instead of having a single pair of overtwisted disks, we require two pairs of overtwisted disks, each pair in the image of the other through the \mathbb{Z}_2 -action.
- (iv) Eliashberg's construction is not \mathbb{Z}_2 -invariant. Therefore we proceed by quotienting the whole manifold by the \mathbb{Z}_2 -action, we then obtain an almost contact pencil over the quotient. The fibres are oriented since they are 3-dimensional almost contact manifolds; the induced almost contact distribution on them is non-coorientable. However, there is no hypothesis on the coorientability in the results of [El]. Once the procedure described in Section 6 is applied, we lift again to the cover.
- (v) Section 7 is trivially adapted to the \mathbb{Z}_2 -invariant setting at the price of overloading the notation.
- (vi) Filling the holes slightly changes. We need to produce a \mathbb{Z}_2 -invariant standard model over $M \times \mathbb{S}^2$, with (M, α_0) a contact manifold with a \mathbb{Z}_2 -invariant action. The only needed change is to make sure that the framing $\{\alpha_0, \alpha_1, \alpha_2\}$ is chosen \mathbb{Z}_2 -invariant. The rest of the proof works through up to notation details.
- (vii) Blowing-down is still a \mathbb{Z}_2 -invariant procedure if the previous choices have been done \mathbb{Z}_2 -invariantly. Therefore, we obtain a \mathbb{Z}_2 -invariant contact structure ξ'_2 on M_2 . Its quotient produces a contact structure on M .

This proves the existence part of the statement. The statement concerning the Chern class follows. \square

APPENDIX: GENERAL CASE

The main result of the article is the existence of a contact structure in an almost contact 5-fold M . Further, if there are no 2-torsion elements in the cohomology group $H^2(M, \mathbb{Z})$, the distribution defining the contact structure can be chosen to be homotopic to the initial almost contact distribution. The technical hypothesis regarding the 2-torsion arises for obstruction-theoretic reasons. In the first version of this article we did not provide a method for handling such remaining situation: this is the aim of this appendix. This fits coherently in the current state of the art since J. Etnyre has just informed us of an open book approach to the same result. Let us show that the contact pencil proof can be enhanced to include the 2-torsion case. Thus, the main result of the article is the following

Theorem 0.1. *There exists a contact structure in every homotopy class of almost contact structures in any 5-fold.*

The homotopy class of the almost contact structure is close to be completely determined by the first Chern class. In order to conclude the general result we should study the change of the whole homotopy type in the deformations. There are only two steps in the proof in which there is no control over the homotopy type, these are:

- (i) The deformation produced when filling the holes, which is shown to be a rough homotopy.
- (ii) The blowing-down process in which only the first Chern class is explicitly described.

Let us describe the behaviour of the obstruction class in these two processes. For the former:

Lemma 0.2. *Under the hypotheses of Theorem 9.1, the distribution ξ' is homotopic to the distribution ξ .*

Proof. Let us show that filling one hole does not change the homotopy type of the distribution. It is mere routine to repeat the argument for multiple holes. The hole is diffeomorphic to $F \times D^2$. Let $f_F : F \rightarrow [0, 1]$ be a Morse function on the fiber F , and assume that it has a single minimum $q \in F$. Therefore, the function

$$f = f_F - r^2 : F \times D^2 \rightarrow \mathbb{R}$$

is a Morse function over $F \times D^2$ whose critical points belong to the central fibre. Extend the function to a homonymous Morse function $f : \widetilde{M} \rightarrow \mathbb{R}$. Let us use the associated cell decomposition generated by the descending manifolds associated to each critical point. The 2-cell associated to a critical point p will be denoted σ_p^2 .

After Lemma 9.5, the obstruction class between ξ and ξ' is an element $[d] \in H^2(\widetilde{M}, \mathbb{Z})$. Let us compute a representative d of it as a cellular cochain.

Denote by \widetilde{M}_j the j -skeleton associated to the cell decomposition of \widetilde{M} and $C_2(\widetilde{M}_2, \widetilde{M}_1)$ the free \mathbb{Z} -module generated by the 2-cells. We may now evaluate the obstruction class on the generating 2-cells:

$$\langle d, \sigma_p^2 \rangle = \begin{cases} a, & p = q, \\ 0, & p \neq q. \end{cases}$$

The obstruction class vanishes in the 2-cells different from σ_q^2 because σ_q^2 is the only 2-cell lying on $F \times D^2$, the rest of them are part of the complementary $\widetilde{M} \setminus (F \times D^2)$, where the distributions coincide. Using Lemma 9.5 again, we also know that $2[d] = 0$.

The proof of the main theorem will remain intact except for the following operation. Instead of gluing in the hole the local model provided by Proposition 8.5, we glue the model doubled. More precisely, instead of using the formula

$$\begin{aligned} e_0^{2,k}(t, \theta) &= \cos(2\pi t), \\ e_1^{2,k}(t, \theta) &= \sin(2\pi t) \cos(2k\pi\theta), \\ e_2^{2,k}(t, \theta) &= \sin(2\pi t) \sin(2k\pi\theta), \end{aligned}$$

from page 50, we rotate twice and so it changes by

$$\begin{aligned} e_0^{2,k}(t, \theta) &= \cos(4\pi t), \\ e_1^{2,k}(t, \theta) &= \sin(4\pi t) \cos(2k\pi\theta), \\ e_2^{2,k}(t, \theta) &= \sin(4\pi t) \sin(2k\pi\theta), \end{aligned}$$

to obtain a new distribution ξ'' . It is immediate that the obstruction class between ξ and ξ'' is $d'' = 2d$ and therefore we have that $[d''] = 0$. Hence, both distributions are homotopic through almost contact distributions. Thus, we can glue for each hole this *double full Lutz twist* and therefore the resulting distributions remain homotopic. \square

For the later deformation, the blow-down process, the argument is even simpler:

Lemma 0.3. *Under the hypotheses of Lemma 9.4, the distributions ξ and ξ_1 are homotopic as almost contact distributions.*

Proof. The distributions $\tilde{\xi}$ and $\tilde{\xi}_1$ in the blow-up \widetilde{M} are homotopic after the previous Lemma. Therefore, ξ and ξ_1 are homotopic over $M \setminus B$. Let us consider a cell decomposition of the manifold such that B contains only 4-cells and 5-cells, this is possible because of dimensional reasons. It is then clear that the obstruction d vanishes for that cell decomposition. Geometrically, the obstruction class does not see the blow-down process. \square

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